

Synchronous Orbit of a Satellite about a Binary Star—C.E. Mungan, Spring 2004

Two equal-mass stars circle each other a distance R apart. A much lighter third body is in line with them (but not between them) and orbits them synchronously. What is the radius x of its orbit?

Denoting the mass of each star by M , we apply Newton's second law to obtain

$$\frac{GM^2}{R^2} = M \frac{v^2}{r} \quad (1)$$

where the speed of either star is $v = 2\pi r / T$, with $r = R/2$ the radius of its orbit and T its period, so that

$$T^2 = \frac{2\pi^2 R^3}{GM} \quad (2)$$

which is Kepler's third law for a binary star system. Similarly for the satellite, which is sufficiently light that it does not perturb the circular motion of the stars, Newton's second law becomes

$$\frac{GmM}{(x - R/2)^2} + \frac{GmM}{(x + R/2)^2} = m \frac{(2\pi x / T)^2}{x} \quad (3)$$

Substituting Eq. (2) into (3) and rearranging (by bringing the LHS to a common denominator and then cross-multiplying) gives

$$z^5 = \frac{1}{2}z^3 + z^2 - \frac{1}{16}z + \frac{1}{4} \quad (4)$$

where $z \equiv x / R$. This can be iterated: start with say an initial guess of $z = 1$ on the RHS, to find $z = 1.11$ on the LHS, then insert that back into the RHS, and so on to quickly converge to the solution of $x \approx 1.2R$.

A couple of closing comments on this problem are in order:

1. The result is independent of the (common) mass of the binary stars. The reason for this is that the gravitational acceleration of both one star and of the satellite are proportional to the mass of the other star. The situation is akin to Galileo's mythical experiment wherein dropped cannon balls of different masses fall with the same acceleration.
2. What we have calculated here is the second/third Lagrange point (well known to be unstable) for an equal-mass astronomical system. The other two independent points are L1 (also unstable) between the stars (specifically, trivially midway between them at the center of mass) and L4/5 (stable) at the vertices of the two equilateral triangles with sides of length R whose other vertices are at the two stars. It is left as an easy exercise for the reader to verify that such points L4/5 are indeed synchronous.