Review of the Brachistochrone Problem—C.E. Mungan, Fall 2012

Here I review the derivation of some key results about the curve of shortest descent time for a bead sliding on a frictionless wire (starting from rest) in a vertical plane connecting the origin to the point (x, y) in the first quadrant, where the *x*-axis points positive to the right and the *y*-axis points positive downward. The first few steps follow the method of LAJPE **6**, 196 (2012),

From conservation of mechanical energy, we can find the speed of the bead at any point on the wire as

$$\tfrac{1}{2}m\upsilon^2 = mgy \quad \Rightarrow \quad \upsilon = \sqrt{2gy} \tag{1}$$

where *m* is the mass of the bead and *g* is earth's gravitational field strength. Since the curve is confined to the *xy*-plane, its differential arclength is

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + (dy/dx)^2}\, dx\ . \tag{2}$$

Since $\upsilon = ds/dt$, the total descent time to the endpoint (x, y) is

$$T = (2g)^{-1/2} \int \sqrt{\frac{1 + y'^2}{y}}\, dx\ . \tag{3}$$

Writing the integrand as $f(y, y')$, the descent time is minimized if *f* satisfies the Euler-Lagrange equation

$$\frac{\partial f}{\partial y} = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right). \tag{4}$$

Since *f* does not contain *x* explicitly, it satisfies the Beltrami identity derived in the Appendix,

$$f - y'\frac{\partial f}{\partial y'} = \text{constant} \tag{5}$$

where we will choose to write the constant as $(2a)^{-1/2}$. Evaluating this identity for the integrand of Eq. (3) results in

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right] y = 2a\ . \tag{6}$$

This result can be separated and integrated, but that is not necessary. Writing $dy/dx = (dy/d\phi)/(dx/d\phi)$, one can directly verify that the parametric solution of Eq. (6) that passes through the origin is

$$x = a(\phi - \sin\phi) \quad \text{and} \quad y = a(1 - \cos\phi) \tag{7}$$

where $a > 0$ and $0 \le \phi \le 2\pi$ are the radius and rolling angle of a wheel tracing out an inverted cycloid. Alternatively we can transform these rectangular coordinates into polar coordinates

(r,θ) given by

$$\left(\frac{r}{a}\right)^2 = \phi(\phi - 2\sin\phi) + 2(1-\cos\phi) \quad \text{and} \quad \tan\theta = \frac{1-\cos\phi}{\phi - \sin\phi}\,. \tag{8}$$

Given an endpoint (r,θ) in the first quadrant, one can numerically compute the parametric coordinates as follows. First find ϕ from θ using the second equality in Eq. (8); there is a unique solution since θ monotonically decreases from $\pi/2$ to 0 as ϕ increases from 0 to 2π. Then substitute that value of ϕ along with r into the first equality in Eq. (8) to find a. Two special cases are:

(i) The bottom point of the cycloid when the wheel has rolled halfway around so that $\phi = \pi$, in which case $x = \pi a$ and $y = 2a$, so that $\theta = \tan^{-1}(2/\pi) \approx 32.5°$ and $a = r(\pi^2+4)^{-1/2} \approx 0.269r$.

(ii) The end cusp of the cycloid when the wheel has rolling all the way around so that $\phi = 2\pi$, in which case $x = 2\pi a$ and $y = 0$, such that $\theta = 0$ and $a = r/2\pi \approx 0.159r$.

Now let's go back and actually find the descent time. Using Eq. (7) we can write

$$y' = \frac{\sin\phi}{1-\cos\phi} = \sqrt{\frac{1+\cos\phi}{1-\cos\phi}} = \sqrt{\frac{2a-y}{y}}\,. \tag{9}$$

Also we can multiply and divide by dy underneath the integral sign in Eq. (3) to get

$$T = (2g)^{-1/2}\int\sqrt{\frac{1+1/y'^2}{y}}\,dy\,. \tag{10}$$

Substituting Eq. (9) into (10) gives

$$T = \sqrt{\frac{a}{g}}\int\frac{dy}{\sqrt{y(2a-y)}} = 2\sqrt{\frac{a}{g}}\tan^{-1}\sqrt{\frac{y}{2a-y}} \tag{11}$$

which is one way to write the result. A more compact form is obtained by substituting Eq. (7) into (11) to get

$$T = 2\sqrt{\frac{a}{g}}\tan^{-1}\sqrt{\frac{1-\cos\phi}{1+\cos\phi}} = \phi\sqrt{\frac{a}{g}}\,. \tag{12}$$

The last step follows from the identities

$$\sin^2\left(\frac{\phi}{2}\right) = \frac{1-\cos\phi}{2} \quad \text{and} \quad \cos^2\left(\frac{\phi}{2}\right) = \frac{1+\cos\phi}{2}\,. \tag{13}$$

We can check Eq. (12) for the two special cases mentioned above. At the lowest point on the cycloid when $y = 2a$, T equals the *isochronous* time

$$\tau = \pi\sqrt{\frac{a}{g}}\,. \tag{14}$$

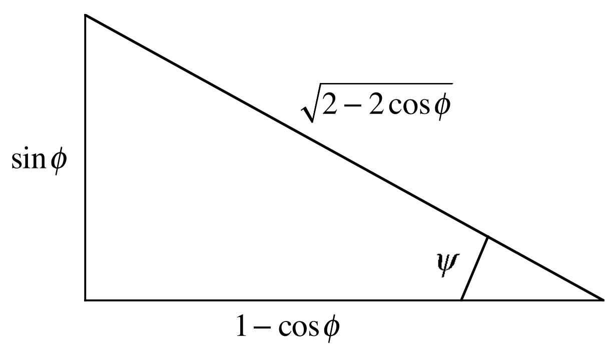
If the bead were to start from rest at *any* point along the cycloid (not necessarily at the origin), the time it would take to slide down to the lowest point is always given by Eq. (14). In other words, a cycloid is also the solution to the tautochrone problem—the trajectory that a simple pendulum bob would have to follow so that its period is independent of amplitude even if the angle is not small. An elegant way to prove this result, as remarked in Amer. Math. Monthly **103**, 242 (1996), is to move the origin to the bottom point of the cycloid and show that the restoring force on the bead is simple harmonic. In other words, the distance s that the bead is displaced along the curve is proportional to the component of the gravitational force $-mg\sin\psi$ at that point on the wire (where ψ is the angle that the curve makes relative to the horizontal). The arclength can be obtained from Eq. (7) as

$$s = \int\sqrt{(dx)^2 + (dy)^2} = a\int\sqrt{2-2\cos\phi}\,d\phi = 2a\int\sin(\phi/2)\,d\phi = -4a\cos(\phi/2)\,, \qquad (15)$$

while the tangential restoring force is given from Eq. (9) as

$$\tan\psi = \frac{dy}{dx} = \frac{\sin\phi}{1-\cos\phi} \quad\Rightarrow\quad \sin\psi = \frac{\sin\phi}{\sqrt{2-2\cos\phi}} = \frac{2\cos(\phi/2)\sin(\phi/2)}{2\sin(\phi/2)} = \cos(\phi/2)\,. \qquad (16)$$

Here I translated from tangent to sine by drawing the triangle below with the hypoteneuse calculated using the Pythagoras theorem.



Comparing Eqs. (15) and (16), we see that $s \propto -\sin\psi$ as we wished to show.

To shift the origin back to the left cusp of the cycloid, we can evaluate Eq. (15) between limits of π and $\pi+\phi$ to obtain

$$s = -4a\left[\cos\left(\frac{\pi}{2}+\frac{\phi}{2}\right) - \cos\left(\frac{\pi}{2}\right)\right] = 4a\sin(\phi/2) \qquad (17)$$

for the arclength of our standard cycloid. This result is valid for the first half of the cycloid, over the range $0 \le \phi \le \pi$. It correctly gives zero for $\phi = 0$. When the wheel has rolled halfway around, we have $x = \pi a$, $y = 2a$, and $s = 4a$ which is a bit larger than $(\pi^2 + 2^2)^{1/2}a$, as expected since the cycloid is a curved arc and not a straight line.

Appendix: Derivation of the Beltrami identity

If f is a function only of y and y', its derivative is

$$\frac{df}{dx} = \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \qquad \text{(A.1)}$$

where primes refer to total x-derivatives. Substituting Eq. (4) into the first term on the right-hand side gives

$$\frac{df}{dx} = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) y' + \frac{\partial f}{\partial y'} \frac{dy'}{dx} = \frac{d}{dx}\left(\frac{\partial f}{\partial y'} y'\right) \qquad \text{(A.2)}$$

Integrating both sides with respect to x results in Eq. (5), as we wanted to show.