Uniqueness of Brachistochrone Solution—C.E. Mungan, Spring 2017

This document proves that a cycloid (as opposed to some other shape such as a parabola) is the *unique* analytic solution to the brachistochrone problem.

Define $T_{\text{cyc}}(x,y)$ to be the time to get from (0,0) to (x,y) along a cycloid that starts out with initially vertically downward slope. (Comment: There is only *one* such cycloid that passes through given initial and final points with given initial slope. An initially vertical slope, corresponding to a cyloid whose cusp is at the initial point, corresponds to a sliding particle that starts from rest. In other words, if we think of the usual rolling-wheel method of drawing a cycloid, there is a unique radius R of that wheel and the wheel must start at zero angle θ with the rim point of interest in contact with the ground.) Now observe that the cycloidal descent time from (0,0) to (X,Y) can be written in terms of an integral along an *arbitrary* path from (0,0) to (X,Y) as

$$T_{\text{cyc}} = \int_{\text{arb}} dT_{\text{cyc}} \,. \tag{1}$$

The reasoning here is that we are doing the integral of a perfect differential, and so the answer depends *only* on the two endpoints (0,0) and (X,Y) which are identical both for the arbitrary path and for the cycloidal path. However, we can rewrite the differential as the dot product of a gradient with the differential line element,

$$dT_{\text{cyc}} = \vec{\nabla} T_{\text{cyc}} \cdot d\vec{s} \,. \tag{2}$$

To prove this equality, simply write out the components of each vector, take the dot product, and then you will recognize the result as Taylor's theorem in the infinitesimal limit.

It now follows that

$$T_{\text{cyc}} = \int_{\text{arb}} \vec{\nabla} T_{\text{cyc}} \cdot d\vec{s} \le \int_{\text{arb}} \left| \vec{\nabla} T_{\text{cyc}} \right| ds \tag{3}$$

where the last inequality follows from the fact that we dropped the cosine of the angle between the two vectors in the dot product. (The cosine of any angle is always less than or equal to unity.) However, as proven in Appendix A of EJP **34**:59 (2013),

$$\left| \vec{\nabla} T_{\text{cyc}} \right| = \frac{1}{\upsilon} \tag{4}$$

for frictionless descent of a particle along a cycloid starting from rest at the origin, where $\upsilon = \sqrt{2gy}$ is the speed after descent by a vertical distance of y from energy conservation. Using this result and the fact that $\upsilon \equiv ds/dt$, Eq. (3) becomes

$$T_{\text{cyc}} \leq \int_{\text{arb}} dt = T_{\text{arb}} \, . \tag{5}$$

This inequality is what we set out to prove: the cycloid gives a descent time that is smaller than the descent time for any other arbitrary curve. (Comment: Equality holds only if the previously mentioned cosine always equals unity, which only happens if you move along a cycloid so that $\vec{\nabla} T_{\text{cyc}}$ is always parallel to $d\vec{s}$.)

Note that this proof does not rule out nonanalytic solutions. For example, one can splice together two different curves that together give the same descent time as does a single cycloid, as shown in AJP **84**:917 (2016). There is a discontinuity in the gradient at the splice point, however, which invalidates the proof presented here.