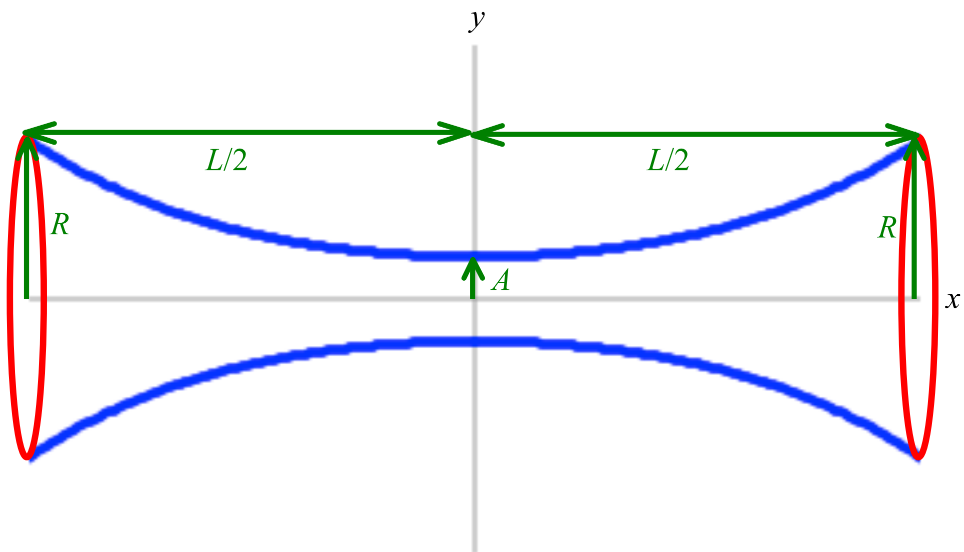
Breaking of a Cylindrical Soap Film—C.E. Mungan, Fall 2019

Two coaxial circular rings of radius R are separated by a distance of L. They are dipped into a soap bubble solution to form a soap film between the rings that is a surface of revolution, as sketched below, formed by rotating the blue curve $y(x)$ about the x axis of symmetry. The y axis is chosen to intersect the point that the film approaches closest to the x axis, so that $y = A$ say with $dy/dx = 0$ at that point. What is the maximum value of L in terms of R, beyond which the film breaks?



This problem can be solved in two steps. First we derive a formula for $y(x)$ using the calculus of variations and fit its parameters to the preceding geometry. Then we find the value of L for which the film breaks.

Ignoring gravity, the only force acting on a patch of the soap film is surface tension, which acts to minimize the area of the entire film (because surface tension is the elastic potential energy per unit surface area). Following the solution to problem 6.19 in Taylor's *Classical Mechanics*, the area S of a surface of revolution for a curve of arclength s is

$$S = \int 2\pi y\, ds = 2\pi \int y\sqrt{1+x'^2}\, dy = 2\pi \int f\, dy \tag{1}$$

where $x' = dx/dy$. Here $f(x,y,x')$ has no explicit dependence on x, so that the Euler-Lagrange equation becomes $\partial f/\partial x' = \text{constant}$ where we call the constant A. That result becomes

$$\frac{yx'}{\sqrt{1+x'^2}} = A \quad \Rightarrow \quad x' = \frac{A}{\sqrt{y^2 - A^2}}. \tag{2}$$

An alternative method of deriving Eq. (2) without using the calculus of variations is provided in the Appendix. Using the substitution $y = A\cosh u$, the solution to this differential equation is found to be

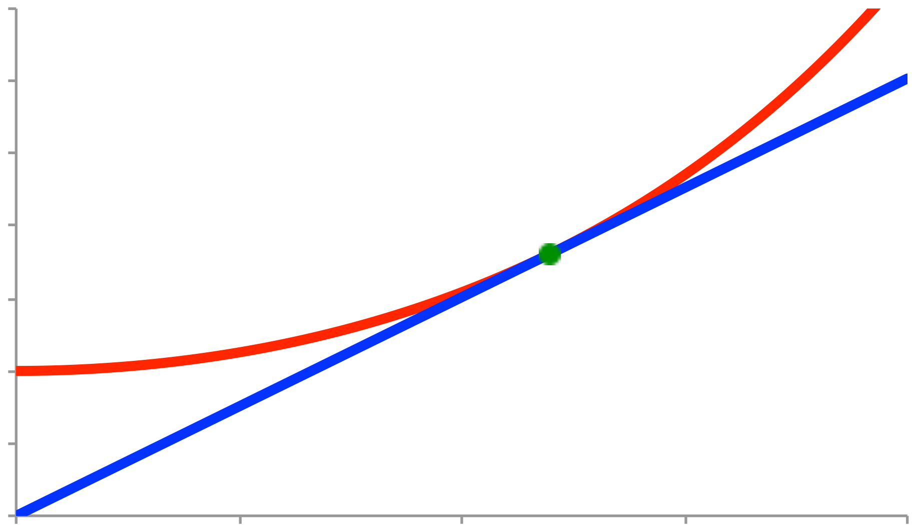
$$y = A\cosh\frac{x}{A} \tag{3}$$

for the boundary conditions $y = A$ and $dy/dx = 0$ at $x = 0$. To find the value of A, we require the film to intersect the ring so that

$$R = A\cosh\frac{L}{2A} \quad \Rightarrow \quad \frac{2R}{L}z = \cosh z \tag{4}$$

in terms of the dimensionless parameter $z \equiv L/2A$. That completes the first step.

For the second step, the left and right sides of Eq. (4) are plotted versus z in blue and red, respectively, in the following graph.



We see that they just cease to have an intersection for large L when their slopes at the point z of solution (indicated by the green dot) are equal to each other, so that

$$\frac{2R}{L} = \sinh z \tag{5}$$

by taking the derivatives of both sides of Eq. (4). Dividing Eq. (4) by (5) implies that

$$z = \coth z \tag{6}$$

whose numerical root is $z \approx 1.19968$. If we subtract the square of Eq. (5) from the square of Eq. (4), we find

$$\frac{L}{R} = 2\sqrt{z^2 - 1} \quad \Rightarrow \quad \boxed{L_{\max} \approx 1.32549R}\,. \tag{7}$$

Surprisingly, the film does not break when $A \approx 0$ but instead when $A = R\sqrt{1 - 1/z^2} \approx 0.55243R$. We can understand this behavior by plotting the surface area S by substituting Eq. (3) into (1) to get

$$S = 2\pi A \int_{-L/2}^{L/2} \cosh^2\frac{x}{A}\,dx = 4\pi A^2 \int_0^z \cosh^2 u\,du \tag{8}$$

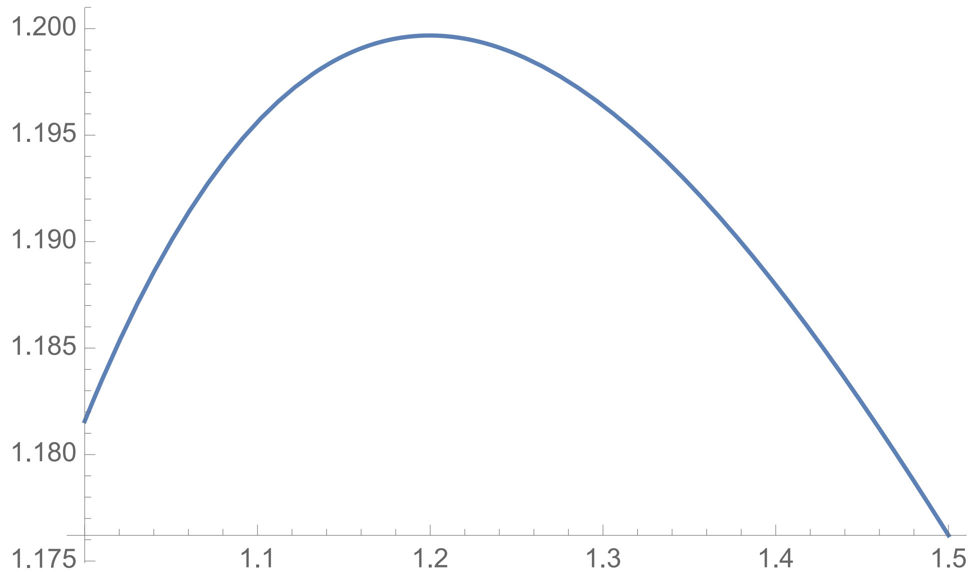
where $u \equiv x/A$. Using the hyperbolic identities $\cosh 2u = 2\cosh^2 u - 1$ and $\sinh 2z = 2\cosh z \sinh z$, the integral becomes

$$S = 2\pi A^2\left(z + \cosh z \sinh z\right) \quad \Rightarrow \quad \frac{S}{2\pi R^2} = \frac{A^2}{R^2}\left(z + \cosh z \sinh z\right). \tag{9}$$

Substitute $A/R = \operatorname{sech} z$ from Eq. (4) and note that the surface area of the two rings together is $S_{\min} = 2\pi R^2$ which is the global minimum area that arises when there is a separate film spanning each ring. Then Eq. (9) becomes

$$\frac{S}{S_{\min}} = z \operatorname{sech}^2 z + \tanh z . \tag{10}$$

This area ratio is plotted below versus z. We see that S increases as z increases until we hit the critical value $z \approx 1.19968$ given by the solution of Eq. (6) which is also when the derivative of Eq. (10) becomes equal to zero. (Interestingly, $S/S_{\min}$ is also equal to 1.19968 when z is equal to that value.) Beyond that critical value, the area S of the film will spontaneously slide downhill to the value $S_{\min}$ by breaking into two parts. That is, if L is larger than the value given by Eq. (7), then when the assembly of two coaxial rings is dipped into the soap solution, the film will form across each ring surface separately rather than bridging the space between them.



A Mathematica animation showing the breaking of the soap film as L increases (or R decreases) is at http://demonstrations.wolfram.com/SoapFilmBetweenTwoEqualAndParallelRings/.

Appendix: Alternate Derivation of Equation (2)

Consider again the first diagram in this document. Imagine a red ring of radius A encircling the soap film at $x = 0$. Now imagine a second red ring of radius y encircling the soap film at arbitrary horizontal position x along the axis of symmetry. If the surface tension of the film is γ, then we can balance the horizontal forces due to surface tension on the cylinder of film located between these two new red rings as

$$\gamma 2\pi y \cos\theta = \gamma 2\pi A \cos 0 \qquad \text{(A1)}$$

where θ is the angle that the blue curve makes with respect to the horizontal at position x. Equation (A1) simplifies to

$$y\cos\theta = A\,. \qquad \text{(A2)}$$

Now notice that

$$\cos\theta = \frac{dx}{ds} = \frac{dx}{\sqrt{dx^2 + dy^2}} = \frac{x'}{\sqrt{x'^2 + 1}}\,. \qquad \text{(A3)}$$

Substituting Eq. (A3) into (A2) immediately gives Eq. (2). Thanks to Seth Rittenhouse for mentioning this alternate method to me.