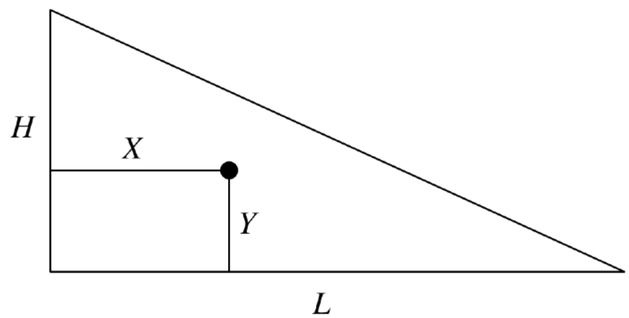
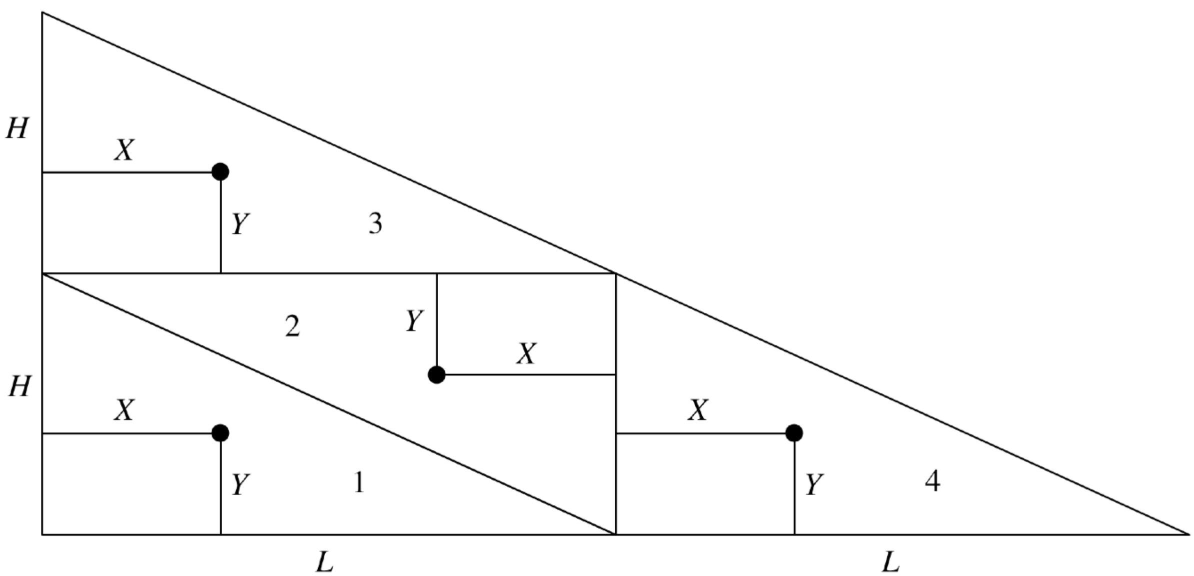
Center of Mass of a Uniform Triangular Plate—C.E. Mungan, Fall 2008

reference: EJP ***29****, 1177 (2008).*

One can find the position (X,Y) of the center of mass (COM) of the right triangular plate of dimensions L and H sketched below without using calculus. Say this plate has mass M.



All one has to do is double the size of the plate and note that it can then be subdivided into four triangles each equal in size to the original triangle above.

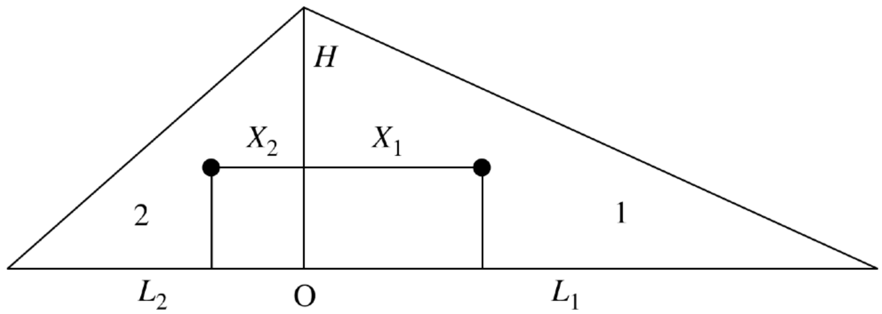


The x position of the COM of this new object can be written in two different ways. First we can say it's the mass-weighted sum of the center of mass, x_i, of each of the four constituent triangles,

$$X_{\text{new}} = \frac{\sum_{i=1}^{4} M x_i}{\sum_{i=1}^{4} M} = \frac{M\left[X + (L - X) + X + (L + X)\right]}{4M}\,. \qquad (1)$$

On the other hand, since the triangle is doubled in size, the COM is twice as far away from the left edge as it used to be, $X_{\text{new}} = 2X$. Equating this to the right-hand side of Eq. (1) and simplifying gives the familiar result, $X = L/3$. In words, the COM is located away from the left edge by one-third of the bottom length of the triangle. Likewise, $Y = H/3$.

We can extend the result to an obtuse triangular plate by dividing it into two right triangles, as sketched below.

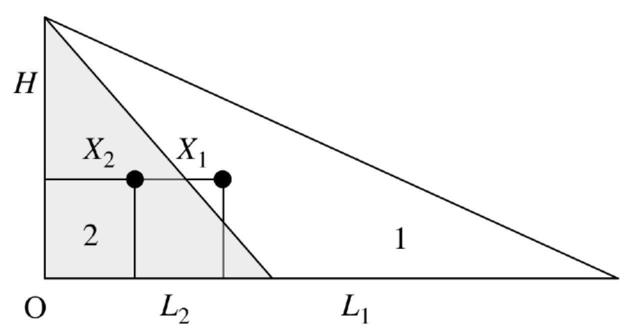


Noting that both triangles have the same height, their individual masses are simply proportional to their horizontal lengths (because the area of triangle *i* is $HL_i/2$). Thus the analog of Eq. (1) becomes

$$X_{\text{obtuse}} = \frac{\sum_{i=1}^{2} L_i x_i}{\sum_{i=1}^{2} L_i} = \frac{L_1 X_1 - L_2 X_2}{L_1 + L_2} = \frac{L_1 - L_2}{3} \tag{2}$$

relative to the indicated origin O. Clearly this result is correct for an isoceles triangle for which $L_1 = L_2$, and it is also correct for a right triangle for which $L_2 = 0$. On the other hand, both constitutent triangles have the *y* position of their center of mass at $H/3$ and hence so does the overall obtuse plate.

Finally consider an acute triangular plate, and again find point O by dropping a perpendicular down from the top vertex. The trick here is to again divide the plate into two right triangles, one of which can be thought of as having negative mass in the sketch below.



In that case we need to modify the denominator of Eq. (2) to get

$$X_{\text{acute}} = \frac{L_1 X_1 - L_2 X_2}{L_1 - L_2} = \frac{L_1 + L_2}{3} . \quad (3)$$

Not surprisingly, the final answer for the x position of the COM in Eq. (3) is the same as that in Eq. (2) with L_2 replaced by $-L_2$, since the obtuse plate has L_2 positive to the left while the acute plate has the opposite sign convention. Note that as $L_2 \to L_1$, X_{acute} approaches the value $2L_1 / 3$. Since $Y_{\text{acute}} = H / 3$, just as for an obtuse plate, the COM then correctly lies along the long right edge of the acute plate. More generally, note that Eq. (3) predicts that the COM always lies within the area of the plate because

$$\frac{2L_1}{3} > \frac{L_1 + L_2}{3} > \frac{2L_2}{3} \quad (4)$$

for any values of L_2 between 0 and L_1.