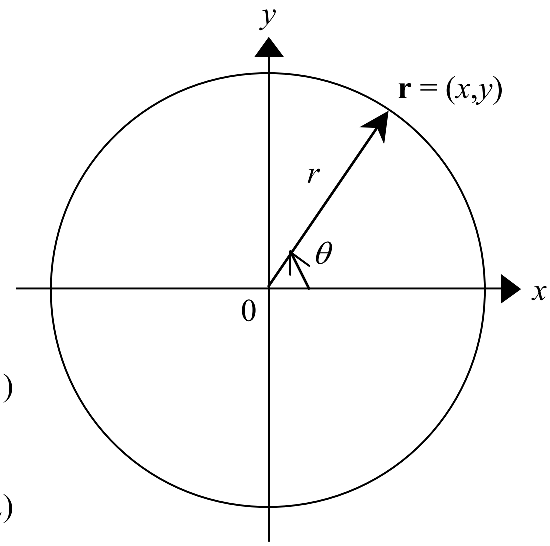
Formal Derivation of Centripetal Acceleration—C.E. Mungan, Summer 2022

Consider a particle executing uniform circular motion (UCM). Place the origin at the center of the circular trajectory of radius r, choose the x-axis to cross the initial position of the particle (i.e, at $t_\text{i} = 0$), and choose the y-axis to point in the direction of the initial velocity of the particle. The fact that the speed υ is constant means that the angle θ that the position vector $\vec{\mathbf{r}}$ makes with the x-axis increases linearly with time, $\theta \propto t$. The constant of proportionality in this equation is called the angular speed ω (with units of rad/s) so that



$$\theta = \omega t \,. \tag{1}$$

The position of the particle at some arbitrary time t is

$$\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} = r\cos\theta\hat{\mathbf{i}} + r\sin\theta\hat{\mathbf{j}} = r\cos\omega t\hat{\mathbf{i}} + r\sin\omega t\hat{\mathbf{j}} \tag{2}$$

using Eq. (1) in the last step. A powerful result you can extract from this is how to express the radial unit vector in Cartesian coordinates,

$$\hat{\mathbf{r}} \equiv \frac{\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|} = \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}} \,. \tag{3}$$

You can check (using Pythagoras' theorem) that the magnitude of this vector is in fact 1.

Next, to find the velocity, differentiate Eq. (2) with respect to time to get

$$\vec{\boldsymbol{\upsilon}} \equiv \frac{d\vec{\mathbf{r}}}{dt} = -r\omega\sin\omega t\hat{\mathbf{i}} + r\omega\cos\omega t\hat{\mathbf{j}} \,. \tag{4}$$

This can be compactly expressed in terms of the azimuthal unit vector $\hat{\boldsymbol{\theta}}$, but that avenue will not be pursued here. Instead, I leave it as another exercise to take the magnitude of Eq. (4) to find the speed,

$$\upsilon = r\omega \,. \tag{5}$$

Finally we get the acceleration by differentiating Eq. (4) with respect to time,

$$\vec{\mathbf{a}} \equiv \frac{d\vec{\boldsymbol{\upsilon}}}{dt} = -r\omega^2\cos\omega t\hat{\mathbf{i}} - r\omega^2\sin\omega t\hat{\mathbf{j}} \,. \tag{6}$$

Comparing this result to Eq. (2), one sees that

$$\vec{\mathbf{a}} = -\omega^2\vec{\mathbf{r}} \tag{7}$$

which implies that the acceleration is radially inward and thus *centripetal* ("center-pointing") with magnitude

$$a = \omega^2 r = \frac{\upsilon^2}{r} \tag{8}$$

using Eq. (5) in the final step. Equation (7) gives one compact vector form for the centripetal acceleration. Another can be obtained from the right-hand screwdriver (RHS) rule: curling one's fingers in the direction of motion, one's thumb points in the direction of the angular velocity vector which is also the direction of the *z*-axis in accord with the right-hand cross-product (RHC) rule for the Cartesian unit vectors, $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$, so that $\vec{\boldsymbol{\omega}} = \omega \hat{\mathbf{k}}$. Now consider the cross-product

$$\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{\upsilon}} = \omega \hat{\mathbf{k}} \times (-r\omega \sin \omega t \hat{\mathbf{i}} + r\omega \cos \omega t \hat{\mathbf{j}}) = -r\omega^2 \sin \omega t \hat{\mathbf{j}} - r\omega^2 \cos \omega t \hat{\mathbf{i}} = \vec{\mathbf{a}} \tag{9}$$

with magnitude $a = \omega \upsilon$.

Uniform circular motion is a key example of non-constant acceleration, so that the standard kinematic equations do not apply. One can use the preceding results to verify that the jerk is

$$\frac{d\vec{\mathbf{a}}}{dt} = r\omega^3 \sin \omega t \hat{\mathbf{i}} - r\omega^3 \cos \omega t \hat{\mathbf{j}} = -\omega^2 \vec{\boldsymbol{\upsilon}} \tag{10}$$

and the snap is

$$\frac{d^2\vec{\mathbf{a}}}{dt^2} = r\omega^4 \cos \omega t \hat{\mathbf{i}} + r\omega^4 \sin \omega t \hat{\mathbf{j}} = \omega^4 \vec{\mathbf{r}}\,. \tag{11}$$

The pattern for the crackle, pop, and higher derivatives [1] of the acceleration are now clear. Another example of non-constant acceleration is simple harmonic motion (SHM) which will follow a similar pattern of higher derivatives as those just found for UCM.

In more advanced treatments, the above derivations can be generalized. First we can loosen the restriction that the speed be constant by allowing ω to be a function of time. (Its time derivative is called the angular acceleration α and is related to the tangential acceleration.) Finally, we can further generalize to arbitrary motion (not necessarily circular) by also allowing r to be a function of time; this is one nice method of generating expressions for the centrifugal and Coriolis accelerations in rotating frames of reference.

[1] https://en.wikipedia.org/wiki/Fourth,_fifth,_and_sixth_derivatives_of_position