Squares on a Chessboard—C.E. Mungan, Summer 2020

How many squares (of all sizes) are there on a chessboard? How many of them contain an equal number of white and black spaces? For generality, suppose the chessboard has \(N\) spaces on a side, where \(N\) is a positive integer.

A standard chessboard has \(N = 8\). There are \(8 \times 8 = 64\) squares of size \(1 \times 1\). For larger squares, set up a coordinate system at the center of the upper left space. Number that space as \((1,1)\). The centers of all other spaces are then of the form \((i,j)\) where \(i\) and \(j\) can run from 1 to 8. For squares of size \(2 \times 2\), we can start at any position \((i,j)\) where \(i\) and \(j\) can run from 1 to 7. Thus there are \(7 \times 7 = 49\) squares of size \(2 \times 2\). We can likewise see that in general the number of squares of size \(n \times n\) is \((9-n) \times (9-n) = (9-n)^2\) where \(n\) runs from 1 to 8. Alternatively we can say the number of squares of size \((9-n) \times (9-n)\) is \(n \times n = n^2\).

Thus for a board with \(N\) spaces on a side, the total number of squares of all sizes is

\[
\sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}
\]

from the standard formula for sums of consecutive squares. For a standard chessboard, that means there are 204 squares on it. Squares of size \(n \times n\) have an equal number of white and black spaces if and only if \(n\) is an even number. That is, the number of squares on the board with an unequal number of white and black spaces is

\[
\sum_{i=1}^{N/2} (2i)^2 = \frac{2^2 (N/2)(N/2+1)(N+1)}{6} = \frac{N(N+1)(N+2)}{6}.
\]

For a standard chessboard, there are thus 120 squares that have an unequal number of white and black spaces. The remainder must have an equal number, for a total equal to the difference between Eqs. (1) and (2) given by

\[
\sum_{i=1}^{N} i^2 - \sum_{i=1}^{N/2} (2i)^2 = \frac{(N-1)N(N+1)}{6}
\]

or 84 for a standard chessboard.