Squares on a Chessboard—C.E. Mungan, Summer 2020

How many squares (of all sizes) are there on a chessboard? How many of them contain an equal number of white and black spaces? For generality, suppose the chessboard has N spaces on a side, where N is a positive integer.

A standard chessboard has $N = 8$. There are $8 \times 8 = 64$ squares of size 1×1. For larger squares, set up a coordinate system at the center of the upper left space. Number that space as $(1,1)$. The centers of all other spaces are then of the form (i, j) where i and j can run from 1 to 8. For squares of size 2×2, we can start at any position (i, j) where i and j can run from 1 to 7. Thus there are $7 \times 7 = 49$ squares of size 2×2. We can likewise see that in general the number of squares of size $n \times n$ is $(9-n) \times (9-n) = (9-n)^2$ where n runs from 1 to 8. Alternatively we can say the number of squares of size $(9-n) \times (9-n)$ is $n \times n = n^2$.

Thus for a board with N spaces on a side, the total number of squares of all sizes is

$$\sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6} \tag{1}$$

from the standard formula for sums of consecutive squares. For a standard chessboard, that means there are 204 squares on it. Squares of size $n \times n$ have an equal number of white and black spaces if and only if n is an even number. That is, the number of squares on the board with an *unequal* number of white and black spaces is

$$\sum_{i=1}^{N/2} (2i)^2 = 2^2 \frac{(N/2)(N/2+1)(N+1)}{6} = \frac{N(N+1)(N+2)}{6}. \tag{2}$$

For a standard chessboard, there are thus 120 squares that have an unequal number of white and black spaces. The remainder must have an equal number, for a total equal to the difference between Eqs. (1) and (2) given by

$$\sum_{i=1}^{N} i^2 - \sum_{i=1}^{N/2} (2i)^2 = \frac{(N-1)N(N+1)}{6} \tag{3}$$

or 84 for a standard chessboard.