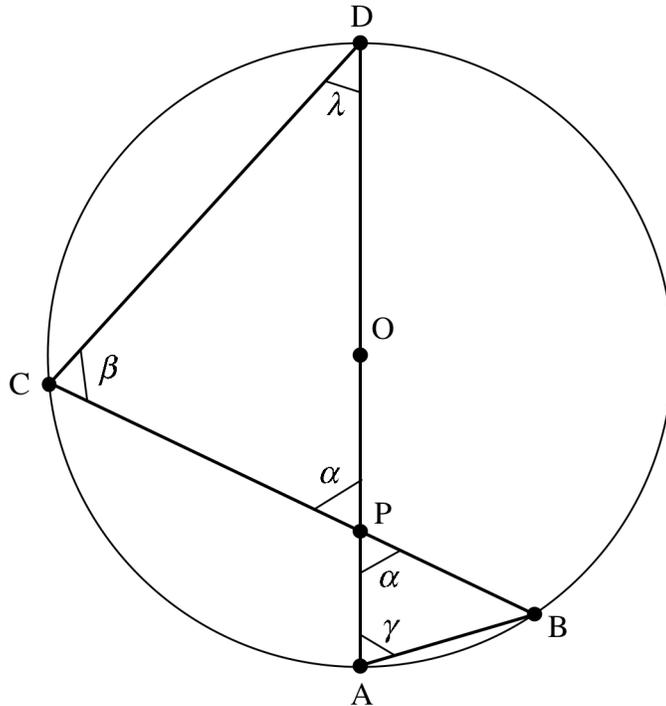


Chords of a Circle—C.E. Mungan, Fall 2007

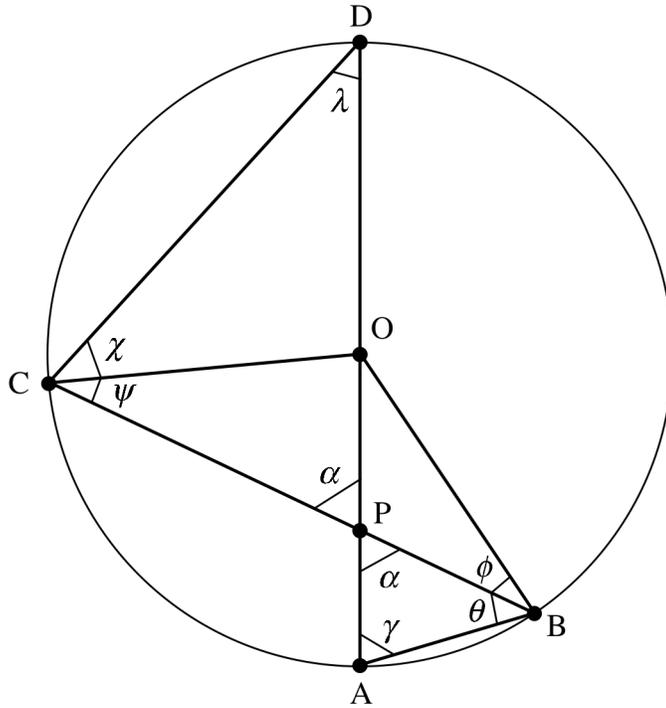
Draw a circle. Now choose any point inside the circle and label it P. Then draw any chord that passes through point P. Point P divides the chord into two lengths. Prove that the product of those two lengths is constant. That is, you would get the same value for the product for any other chord you drew through point P.¹

We prove this by considering two chords that pass through point P. Without loss of generality, we can take one chord to pass through the center O of the circle, as sketched below.



We wish to prove that $AP \cdot PD = BP \cdot PC$. We can see from the above diagram that this will follow if we can prove that triangles APB and PCD are similar, which in turn requires that two pairs of their angles be equal. Clearly, the two angles labeled α are equal, so we will be done if we can show that $\beta = \gamma$, which we can do by drawing in lines OC and BO to get the next diagram.

¹Eli Maor, *The Pythagorean Theorem : A 4000-Year History* (Princeton University Press, 2007).

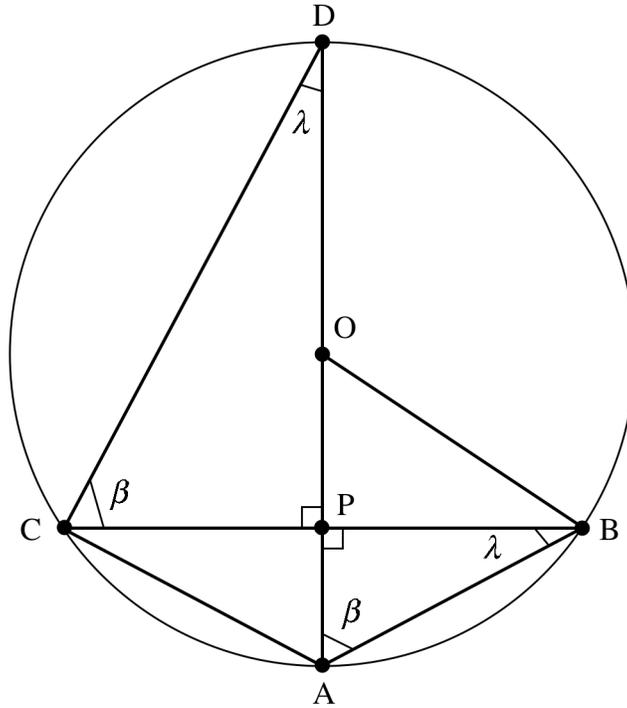


We see from triangle ABP that $\theta = 180^\circ - \alpha - \gamma$. Furthermore triangle ABO is isocetes and therefore $\gamma = \theta + \phi$. Combining these two equations, we conclude that $\phi = 2\gamma + \alpha - 180^\circ$.

By examining triangle CDP in the first figure we see that $\lambda = 180^\circ - \beta - \alpha$. But since triangle CDO in the second figure is isocetes, we conclude that this angle is also equal to χ . Finally, since $\chi + \psi = \beta$, it follows that $\psi = 2\beta + \alpha - 180^\circ$.

Finally, triangle BCO is isocetes and therefore $\phi = \psi$. Equating the boxed equations at the ends of the preceding two paragraphs, we conclude that $\beta = \gamma$, as we wanted to show. (As a check, note that this conclusion also implies that $\theta = \lambda$ using the first equations in each of these two paragraphs.)

We can use this result to prove two interesting corollaries. At the top of the next page, I have redrawn the first figure for the special case of $\alpha = 90^\circ$ and used the same label for all angles that we have already proved to be equal. We see from this third figure that angle ACP is equal to angle ABP, that is to λ . We also notice that β and λ are complementary angles. We therefore conclude that angle ACD is 90 degrees, thereby establishing the first corollary: **If two lines (in a fixed plane) that pass through two fixed points A and D intersect perpendicularly at C, then the locus of points traced out by all such intersection points C is a circle.** Namely it is the circle with diameter AD that passes through points A and D.



As a second corollary, we can derive the equation of a circle or equivalently prove the Pythagoras theorem. Referring to the figure above, let's define lengths $x \equiv OP$ and $y \equiv BP = CP$. Implicitly we have defined a circle in our proof of the chords relationship to be the locus of points equidistant from a fixed point O . Call that distance the radius R . Consequently we conclude that $OB = R$, $DP = R + x$, and $AP = R - x$ by inspection of the figure above.

Now our chords relationship applied to chords BC and AD is $BP \cdot CP = DP \cdot AP$. Using the equalities given in the preceding paragraph, this relationship can be rewritten as $y^2 = (R + x)(R - x)$, which rearranges into $x^2 + y^2 = R^2$, as we wished to prove.