Clearing the Fence—C.E. Mungan, Summer 2026

What is the minimum speed a soccer ball can be kicked off the ground so that it just clears a fence of vertical height y whose base is a horizontal distance x away?

The equations of kinematics specify that a ball kicked at speed υ and angle θ (relative to the ground) will travel a horizontal distance x in a time t given by

$$x = (\upsilon\cos\theta)t \quad \Rightarrow \quad t = \frac{x}{\upsilon\cos\theta} \tag{1}$$

and a vertical distance of

$$y = (\upsilon\sin\theta)t - \tfrac{1}{2}gt^2 \,. \tag{2}$$

Substituting Eq. (1) into (2) leads to

$$\upsilon = \sqrt{\frac{gx^2/2}{x\sin\theta\cos\theta - y\cos^2\theta}} \,. \tag{3}$$

To minimize υ, the denominator is maximized with respect to θ to get

$$\frac{d}{d\theta}\left(x\sin\theta\cos\theta - y\cos^2\theta\right) = 0 \quad \Rightarrow \quad \tan 2\theta = -\frac{x}{y} \,. \tag{4}$$

A calculator will return a negative value for 2θ and so 180° is added to it to find the optimal kicking angle of

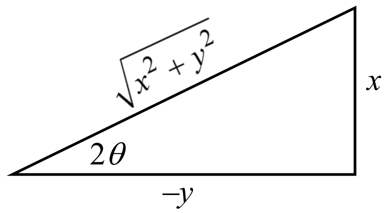
$$\boxed{\theta = 90^\circ - \frac{1}{2}\tan^{-1}\frac{x}{y}} \,. \tag{5}$$

As checks, this equation correctly implies that $\theta = 45^\circ$ if $y = 0$, and $\theta = 90^\circ$ if $x = 0$.

Rewriting Eq. (3) as

$$\upsilon = \sqrt{\frac{gx^2}{x\sin 2\theta - y(1+\cos 2\theta)}} \tag{6}$$

the following triangle diagram from Eq. (4) can be used to find the two trig functions in (6).



Thus Eq. (6) becomes

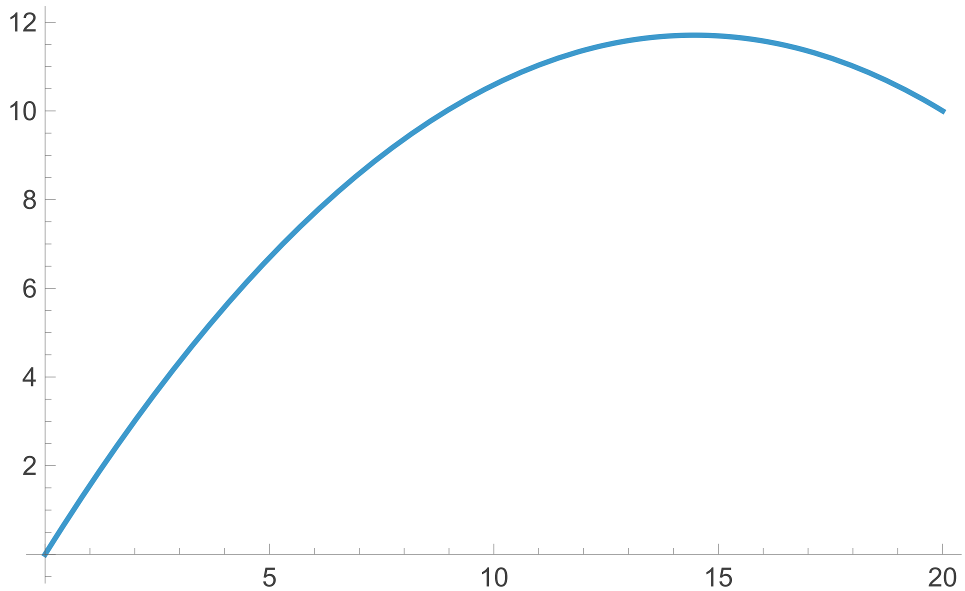
$$\boxed{\upsilon = \sqrt{\frac{gx^2}{(x^2+y^2)^{1/2} - y}}} \,. \tag{7}$$

Making the same checks again, if $y = 0$ the result is $\upsilon = \sqrt{gx}$, consistent with the range formula $R = (\upsilon^2 \sin 2\theta)/g$ when $x = R$ and $\theta = 45°$. For the case of $x \to 0$ the binomial approximation gives

$$\sqrt{x^2 + y^2} = y\left(1 + \frac{x^2}{y^2}\right)^{1/2} \approx y + \frac{x^2}{2y} \tag{8}$$

so that Eq. (7) becomes $\upsilon = \sqrt{2gy}$, consistent with conservation of mechanical energy.

As a more realistic example, if $x = 20$ m and $y = 10$ m then $\theta = 58.3°$ and $\upsilon = 17.8$ m/s $= 39.8$ mph. The trajectory of the ball in space (in meters) is shown below starting at the ground and ending at the top of the fence.



Interestingly, the ball passes its peak before it crosses the fence for *any* values of x and y.

An alternative form of the solution is derived in TPT **55**, 528 (2017). Interested readers may wish to prove that Eq. (5) can be rewritten as

$$\boxed{\theta = 45° + \frac{1}{2}\tan^{-1}\frac{y}{x}} \tag{9}$$

and (7) as

$$\boxed{\upsilon = \sqrt{g\left[y + (x^2 + y^2)^{1/2}\right]}} \tag{10}$$

in accordance with that article.