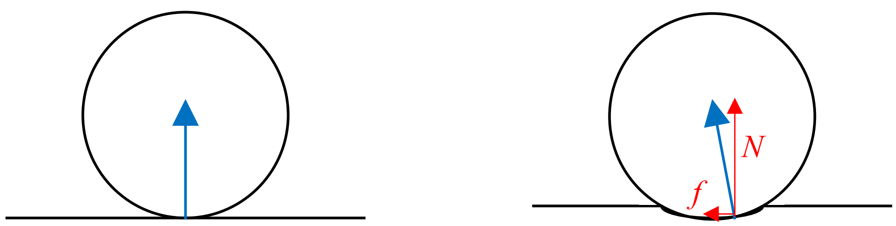
Coefficient of Rolling Friction—C.E. Mungan, Fall 2022

A recent exchange of letters between Cross and Minkin & Sikes (AJP September 2022 pages 649–650) points out that there is more than one way to define the coefficient of rolling friction μ. Consider an unpowered ball or wheel rolling rightward across a horizontal surface. If the ball is perfectly round and rigid, and the surface is perfectly flat and rigid, then there is contact between them at only one point (or only along a line perpendicular to the plane of motion in the case of a wheel). As sketched in panel (a) of the following figure, that means the contact force must be perpendicular to the surface and hence can be denoted as the normal force N. It cannot have a component parallel to the surface, because a backward component would increase the angular speed of the unpowered ball, whereas a forward component would increase the translational speed. (Hence an object can "roll without slipping" in the absence of friction, say by hitting a light-weight ball with a pool cue above center on an air hockey table so that it ends up with just the right top spin such that the ratio of its linear and angular speeds equals the radius of the ball.) On the other hand, if the ball and/or surface are deformable (or there are irregularities on the ball or surface, and there could be adhesion between the ball and surface that needs to be torn apart on the back edge of the wheel as it separates from the road) then the net contact force would shift forward of the center of the ball and tilt slightly backward from the vertical, as shown in panel (b). In that case, one can decompose the contact force into a vertically upward normal force N and a horizontally backward frictional force f.



(a) rigid (b) deformed

The first definition of the coefficient of rolling friction is

$$\boxed{\mu \equiv \frac{f}{N}}. \tag{1}$$

From panel (b), we see that the normal force must balance the object's weight,

$$N = mg \tag{2}$$

where m is the mass of the ball and g is earth's surface gravitational field strength, and thus the friction force must be

$$f = \mu mg \tag{3}$$

according to Eq. (1). The second definition of the coefficient of rolling friction (distinguished from the first definition with a prime) is

$$\boxed{\mu' \equiv \frac{\tau_{\text{loss}} / R}{N}} \tag{4}$$

where τ_{loss} is the torque that produces a loss of mechanical energy E_{mech}. That torque has been scaled by the radius R of the rolling object to make μ' dimensionless. It is important to realize that the frictional force f does *not* produce any loss of mechanical energy; it merely converts translational kinetic energy K_{tr} into rotational kinetic energy K_{rot} as will be shown below. Since there is no change in gravitational potential energy for a horizontal surface, the mechanical energy equals the sum of these two forms of kinetic energy. The loss in mechanical energy equals the gain in thermal energy E_{therm} of the object and surface.

Let's examine the implications of these definitions from two different points of view: Newton's second law (N2L) and the work-kinetic-energy theorem (W-K) where "work" here refers to center-of-mass work and not to thermodynamic work. Both N2L and W-K will be explored for translations and rotations separately.

Choosing the center-of-mass acceleration a to be signed (unlike the force magnitudes f and N) then N2L for translations in the forward direction becomes

$$ma = -f = -\mu mg \quad \Rightarrow \quad a = -\mu g \tag{5}$$

using Eq. (3). On the other hand, N2L for rotations involves two torques about the center of the rolling object: a positive (clockwise) torque due to friction (with a moment arm approximately equal to R) and a negative (loss) torque due to the normal force with a moment arm equal to D which is the horizontal distance between the center of the ball and the point of application of the contact force in panel (b) above, so that

$$I\alpha = fR - ND\,. \tag{6}$$

Here the moment of inertia of the object is $I = \gamma mR^2$ where the shape factor is $\gamma = 2/5$ for a solid ball, $\gamma = 2/3$ for a hollow ball, $\gamma = 1/2$ for a solid cylinder, and $\gamma = 1$ for a hollow hoop. If the object rolls without slipping then $\alpha = a/R$. Consequently Eq. (6) becomes

$$\gamma a = \mu g - g\frac{D}{R} \tag{7}$$

using Eqs. (2) and (3). Substituting Eq. (5) into (7) now gives the important result

$$\mu = \frac{D}{(1+\gamma)R} \tag{8}$$

which relates the rolling friction coefficient to the deformation distance. On the other hand, the loss torque in Eq. (6) is $\tau_{\text{loss}} = ND$ so that Eq. (4) gives the slightly different result

$$\mu' = \frac{D}{R}\,. \tag{9}$$

To further understand this difference, apply W-K for translations and Eq. (1) to get the loss in translational kinetic energy as

$$\Delta K_{\text{tr}} = -f\Delta x = -\mu N \Delta x \tag{10}$$

where Δx is the distance that the center of the wheel moves forward. (Alternatively, if we imagined the wheel to be coated in paint, Δx would be the length of the stripe that would be painted onto the road.) In other words, an alternative first definition of the coefficient of rolling friction is

$$\mu = \frac{-\Delta K_{\text{tr}}}{N\Delta x}. \tag{11}$$

On the other hand, W-K for rotations gives the loss in rotational kinetic energy as

$$\Delta K_{\text{rot}} = (fR - ND)\Delta\theta = (\mu NR - ND)\frac{\Delta x}{R} = (\mu - \mu')N\Delta x \tag{12}$$

using Eqs. (1) and (9), where $\Delta\theta$ is the angle that a point on the rim rotates around the axle while the axle translates forward by Δx. One sees from Eqs. (8) and (9) that $\mu' > \mu$ so both translational and rotational kinetic energy is lost by rolling friction. The overall loss in mechanical energy is given by the sum of Eqs. (10) and (12),

$$\Delta E_{\text{mech}} = -\mu' N\Delta x \tag{13}$$

which arises solely from the loss torque *ND* due to the normal force; the frictional pseudowork of magnitude $f\Delta x$ makes a negative contribution in Eq. (10) to reduce the translational kinetic energy, but an equal and opposite positive contribution in Eq. (12) which would increase the rotational kinetic energy in the absence of the loss torque. Equation (13) can be rewritten as

$$\mu' = \frac{-\Delta E_{\text{mech}}}{N\Delta x} \tag{14}$$

for comparison with Eq. (11). This result can alternatively be written as

$$\boxed{\Delta E_{\text{therm}} = \mu' N\Delta x}. \tag{15}$$

This could be taken as the second definition of the coefficient of rolling friction, in lieu of Eq. (4).

To summarize, there are two useful definitions of the coefficient of rolling friction. The first definition focuses on the force that slows down the translational speed of an object rolling across a horizontal surface. It is given by Eq. (1) as the ratio of the parallel to the perpendicular components of the net contact force between the object and the surface. The second definition focuses on the dissipation of mechanical into thermal energy according to Eq. (15). The two definitions give slightly different numerical values: the ratio of the first to the second coefficient is 5/7 for a solid sphere according to Eqs. (8) and (9). Given that the coefficient for objects that roll well is typically on the order of 0.01 or less, this difference hardly matters numerically. However, from a conceptual point of view should we favor one definition over the other? That's a matter of opinion, but if we are going to call it "the coefficient of rolling friction" then it should be analogous to the coefficients of the other two familiar forms of friction, namely kinetic and static friction.

Consideration of kinetic friction does not favor one definition over the other. In the case of a box sliding across a rough horizontal floor, both definitions give *identical* coefficients, because $f / N = \Delta E_{\text{therm}} / N\Delta x$ as the reader is invited to prove. That is probably not surprising because

$\Delta E_{\text{therm}} = -\Delta K_{\text{tr}}$ since there is no rotational motion involved in this case. It is satisfying that the two definitions coincide for this common situation of sliding friction.

What about static friction? We require an example where it slows down an object on a horizontal surface, so consider a box on the flatbed of a truck that is braking. In order to involve the coefficient of static friction, suppose the box is on the verge of slipping (without tilting) across the bed. That coefficient is then given by Eq. (1). However, no thermal energy is generated at the contact between the box and flatbed, so Eq. (15) is not applicable. On this basis, I argue that the coefficient of rolling friction is best defined by Eq. (1) and not by Eqs. (4) or (15). Perhaps μ' should instead be called something like the “coefficient of rolling dissipation.” This suggestion is strengthened by noting again, as discussed above, that it is not the friction force (which is the component of the contact force parallel to the surface) that causes the dissipation of mechanical energy; on the contrary it is the normal force (i.e., the perpendicular component of the off-center contact force) that produces the loss torque responsible for the dissipation. So it would preferable if the name of the *dissipative* coefficient did not include the word *friction* which should instead be associated with the *force* that changes or impedes the translational motion of an object. In this sense I respectfully disagree with Minkin & Sikes who wish to define “friction” as “what causes a dissipation of mechanical energy” rather than more conventionally as a “force between objects in contact.” I also don’t see why they think the force model works only if the rolling object deforms and not if the surface deforms.