Velocity of an Initially Stationary Target after a Projectile Impacts it Head On
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Suppose that a projectile of initial linear momentum P makes a 1D collision with an initially stationary target. Let M be the sum of the masses of the projectile and target. Find the final velocity v of the target if the coefficient of restitution (COR) is e.

Let the projectile have mass m_1, initial velocity u_i, and final velocity u_f. Let the target have mass m_2. Then conservation of linear momentum implies

$$P = m_1 u_f + m_2 v \tag{1}$$

where $P = m_1 u_i$ is the constant total momentum of the system of projectile and target. The COR is defined to be the negative ratio of the relative velocities,

$$e \equiv \frac{v - u_f}{u_i - 0} \quad \Rightarrow \quad u_f = v - \frac{eP}{m_1}. \tag{2}$$

Substituting Eq. (2) into (1) and rearranging gives the final result

$$\boxed{\frac{v - v_{com}}{v_{com}} = e} \tag{3}$$

where $v_{com} \equiv P/M$ is the velocity of the center of mass of the system.

For example, suppose the projectile has mass $m_1 = 5$ kg and velocity $u_i = 4$ m/s, whereas the target has mass $m_2 = 3$ kg. Then, for a perfectly inelastic collision such that $e = 0$, Eq. (3) implies that the final velocity of the target is $v = v_{com} = (5\text{ kg})(4\text{ m/s})/(5\text{ kg} + 3\text{ kg}) = 2.5$ m/s. On the other hand, for a perfectly elastic collision such that $e = 1$, the final velocity of the target is twice as big, $v = 2v_{com} = 5$ m/s. If the system were 50% elastic so that $e = 0.5$, then the target's final velocity would be the average of these two values, namely 3.75 m/s. If the projectile can pass through the target, then e can be negative. For example, if there is a tunnel through the target through which the target can freely pass, then $e = -1$ so that $v = 0$.

The fractional loss in kinetic energy works out to be

$$\frac{K_i - K_f}{K_i} = \frac{m_2(1 - e^2)}{M}. \tag{4}$$

For example, in the perfectly inelastic case, the fractional loss is 3/8 = 37.5% for the numbers in the previous example. Specifically, the initial KE is 40 J and the final KE is 25 J. We see from Eq. (4) that the limits on the COR are $-1 \le e \le 1$, unless there is some stored energy released during the collision such as in the magnetic version of Newton's cradle described in TPT **45**, 409 (2007) and **41**, 158 (2003).