Sums of Consecutive Integers—C.E. Mungan, Spring 2020

Given a positive integer N, in how many different ways can you write it as a sum of two or more consecutive positive integers? For example, 9 can be written exactly 2 ways as $4+5$ and as $2+3+4$, but 8 cannot be so written in any ways.

The answer is equal to the number of odd factors (other than 1) of N. In the preceding examples, 9 has the two odd factors 3 and 9, whereas 8 has no odd factors. In general, if N has no odd factors then it can only have 2 as prime factors. Thus any non-negative integer power of 2 (i.e, 1, 2, 4, 8, …) cannot be written as a sum of consecutive integers.

If N has an odd factor F, then we can write N as the product of the two integers $F \times (N/F)$. For the example of $N = 9$ we have:

- 3×3 which means we add 3 together 3 times as $3+3+3 = (3-1)+(3)+(3+1) = 2+3+4$; and
- 9×1 which means we add 1 together 9 times as

$$1+1+1+1+1+1+1+1+1 =$$

$$(1-4)+(1-3)+(1-2)+(1-1)+(1)+(1+1)+(1+2)+(1+3)+(1+4) =$$

$$(-3-2-1+0+1+2+3)+(4+5) = 4+5.$$

In general if F is a positive odd factor (other than 1) then $M = (F-1)/2$ is a positive integer and we can write $N = F \times (N/F) = (N/F - M) + (N/F - M + 1) + \cdots (N/F + M)$ for a total of $2M + 1 = F$ terms. This is one unique way to write N as a sum of consecutive integers. If the first T terms in this sum are negative, then we can cross off the first $2T+1$ terms in the sum (because they add up to zero) to get a unique sum of consecutive positive integers.

If N can be written as a product of two even factors, no sum can be written from them. For example, $12 = 2 \times 6$ but we cannot convert either $2+2+2+2+2+2$ or $6+6$ into a sum of consecutive integers because there is not a 2 or a 6 at the midpoint of these sums (as there would be if we instead had an odd number of 2's or 6's added together). We want to add 0 to the midpoint integer, ±1 to the next two integers out, ±2 to the next two out from there, and so on just as we did above for the sum of 3's or of 1's.

Finally, if we consider an example such as 15 which has 3 odd factors (namely 3, 5, and 15) then there are indeed 3 ways to write it as a sum of consecutive positive integers:

- 3×5 which means we add 5 together 3 times as $4+5+6$;
- 5×3 which means we add 3 together 5 times as $1+2+3+4+5$; and
- 15×1 which means we add 1 together 15 times as $7+8$ (after crossing off –6 through 6 in the sum)—in general, if N is odd we can write it as $N \times 1$ which reduces to the sum of the two consecutive positive integers $(N-1)/2 + (N+1)/2$.

Note that we do not consider 1×15 because that would mean adding 15 together once, which is not a sum of two or more integers. Also note that if N were 30, we would consider 3×10 but not 10×3. Thus we count only the odd factors (not 10 here), we count every odd factor (both 3×5 and separately 5×3), but we do not count 1.

In case you have doubts about whether you have missed any odd factors, you can start by finding the prime factorization of N, for example $90 = (2)(3)(3)(5)$. Then throw out all factors of 2 to get $\{3,3,5\}$. Finally take all possible products of 1 term at a time, then 2, and so on up to all of the terms. In this example, that gives 3 and 5 taking 1 term, 3×3 and 3×5 taking 2 terms, and $3 \times 3 \times 5$ taking all 3 terms. In other words, the odd factors of 90 are $\{3,5,9,15,45\}$.