Conservation of Relativistic Momentum and Energy—C.E. Mungan, Fall 2019

<u>reference</u>: Taylor "Modern Physics" (2nd edition) problem 2.14

Define the *x*-component of the linear momentum and the total energy of a particle of mass *m* as

$$p_x = m\frac{dx}{d\tau} \quad \text{and} \quad E = mc^2\frac{dt}{d\tau} \tag{1}$$

in the unprimed frame where τ is the proper time, and similarly

$$p'_x = m\frac{dx'}{d\tau} \quad \text{and} \quad E' = mc^2\frac{dt'}{d\tau} \tag{2}$$

in the primed frame. The Lorentz transformation is

$$x' = \gamma\left(x - \upsilon t\right) \quad \text{and} \quad t' = \gamma\left(t - \frac{\upsilon}{c^2}x\right) \tag{3}$$

where it should be remembered that γ and υ are constants. Substitute Eq. (3) into (2) to get

$$p'_x = m\gamma\left(\frac{dx}{d\tau} - \upsilon\frac{dt}{d\tau}\right) \quad \text{and} \quad E' = mc^2\gamma\left(\frac{dt}{d\tau} - \frac{\upsilon}{c^2}\frac{dx}{d\tau}\right). \tag{4}$$

Now replace the four derivatives in these two expressions with those in Eq. (1) to obtain

$$\boxed{p'_x = \gamma\left(p_x - \frac{\upsilon}{c^2}E\right)} \quad \text{and} \quad \boxed{E' = \gamma\left(E - \upsilon p_x\right)}. \tag{5}$$

For the transverse components, since $p_y = m\,dy/d\tau$ and $p'_y = m\,dy'/d\tau$ where $y = y'$ according to the Lorentz transformation, we conclude that $p'_y = p_y$ and likewise $p'_z = p_z$.

Equation (5) has the *same form* as Eq. (3). In other words, if you replace *x* with p_x and *ct* with *E*/*c* (for both the primed and unprimed cases), then the two pairs of equations are identical. The position 4-vector *transforms from one reference frame to another in exactly the same way* that the momentum-energy 4-vector does.

For an *isolated system of particles*, conservation of momentum implies that the initial momentum equals the final momentum in the unprimed reference frame,

$$\sum_{n=1}^{N}\vec{p}_{\text{i}n} = \sum_{m=1}^{M}\vec{p}_{\text{f}m} \tag{6}$$

where there are *N* particles initially and *M* finally. The *x*-component of this equation is

$$\sum p_{\text{i}nx} = \sum p_{\text{f}mx}\,. \tag{7}$$

Similarly, conservation of energy implies

$$\sum E_{\text{i}n} = \sum E_{\text{f}m}\,. \tag{8}$$

The goal is to show that momentum and energy are also conserved in the primed coordinate system. We do that using the preceding results as follows. The first equality in Eq. (5) implies that

$$\sum p'_{\text{in}x} = \gamma \sum p_{\text{in}x} - \gamma \frac{\upsilon}{c^2} \sum E_{\text{in}} \tag{9}$$

and

$$\sum p'_{\text{fm}x} = \gamma \sum p_{\text{fm}x} - \gamma \frac{\upsilon}{c^2} \sum E_{\text{fm}} \,. \tag{10}$$

Now substitute Eqs. (7) and (8) into the right-hand side of Eq. (9) to get

$$\sum p'_{\text{in}x} = \gamma \sum p_{\text{fm}x} - \gamma \frac{\upsilon}{c^2} \sum E_{\text{fm}} \,. \tag{11}$$

But the right-hand sides of Eqs. (10) and (11) are equal, and thus the left-hand sides must also be equal so that

$$\sum p'_{\text{in}x} = \sum p'_{\text{fm}x} \tag{12}$$

which says that the *x*-component of momentum is conserved in the primed frame. Exactly similar arguments can be used to show that the energy is conserved in the primed frame, and it is even easier to show that the *y*-component and *z*-component of momentum is conserved in the primed frame.