**The Relativistic Rocket—C.E. Mungan, Summer 2001**

John Denker posed the following homework exercise on PHYS-L:

> Suppose an interstellar rocket starts from rest and accelerates such that the passengers feel one gee for one year. What is their speed $\upsilon$ at the end of the year?
> Hint: In units where $c$ = 1, one can write $\upsilon = \tanh(\rho)$ where $\rho$ is called the rapidity. Find $\rho$.

The canonical solution can be found at <http://math.ucr.edu/home/baez/physics/rocket.html> but I don't recognize those acceleration equations and am too lazy to look up Misner, Thorne, and Wheeler to find the derivation.

Here is a more direct proof. Consider two inertial frames of reference: E in which an observer is at rest (approximately on Earth, say), and R which is instantaneously comoving with the rocket at any arbitrary instant in time of interest. Suppose the spaceship has (rest) mass $m$ and momentum $p$ measured by E. Let $t$ be the time since the start of the trip as measured by E, and $T$ be the rocket passenger (i.e., proper) time.

The fact that the passengers feel one gee means that the force $F$ measured in frame R is $mg$. However, as proved by French Eq. (7-20), an observer in frame E measures the same (longitudinal) force, despite the fact that she measures neither the same (relativistic) mass nor acceleration of the rocket. Therefore, by Newton's second law,

$$\frac{dp}{dt} \equiv \frac{d}{dt}(\gamma m \upsilon) = F = mg \tag{1}$$

where $\gamma \equiv (1 - \upsilon^2 / c^2)^{-1/2}$ is the usual gamma factor in special relativity. Integrating both sides with respect to $t$ and rearranging, one easily obtains

$$\boxed{\upsilon = \frac{1}{\sqrt{\dfrac{1}{c^2} + \dfrac{1}{(gt)^2}}}}. \tag{2}$$

This has the expected limits: $\upsilon \to gt$ as $t \to 0$, and $\upsilon \to c$ as $t \to \infty$. Putting in $t$ = 1 yr (assuming the year in the problem refers to a year as measured by an Earth observer) and $g$ = 9.8 m/s$^2$ gives $\upsilon = 0.72c$.

Suppose that a year as measured by the rocket passengers is desired instead. Then we have to replace $dt$ by $\gamma\, dT$ in Eq. (1). After a bit of differentiation, one finds that

$$\gamma^2 d\upsilon = g\, dT \tag{3}$$

which after integrating and simplifying gives

$$\boxed{\frac{\upsilon}{c} = \tanh\left(\frac{gT}{c}\right)}. \tag{4}$$

Putting in $T$ = 1 yr gives $\upsilon / c = 0.77 \cong \tanh(1)$. In units where $c$ = 1, Eq. (4) implies that $\rho = gT$, which is a nice way to remember the solution. This proves that rapidity adds linearly in special relativity, just as velocity does in Galilean relativity.