

## Resistivity of Copper—C.E. Mungan, Spring 2001

This document is essentially a summary of Tipler Chap. 27. Specifically, the goal is to derive Ohm's law by developing a microscopic model for the resistivity of a metal as a function of temperature. Following the lead of Tipler, I will make a numerical estimate for copper at room temperature using some simple ideas from modern physics. Amazingly, the only material parameter needed is the atomic number density,  $n_a$ . This can be calculated as the ratio of the mass density to the atomic mass, the latter in turn being equal to the molar mass divided by Avogadro's number,

$$n_a = \frac{\rho_m}{M / N_A} = \frac{(8.93 \text{ g/cm}^3)(6.022 \times 10^{23} / \text{mol})}{63.5 \text{ g/mol}} = 8.47 \times 10^{28} / \text{m}^3. \quad (1)$$

### I. Basic Formula for Resistivity

The definition of resistivity is  $\rho \equiv E / J$ . The current is carried by electrons of number density  $n_e = n_a$  (assuming that each metal atom contributes one conduction electron) and drift speed  $v_d$ , so that  $J = n_e e v_d$ . These electrons are regularly scattered. If we measure time starting from the last scattering event, the velocity of an electron at any instant is

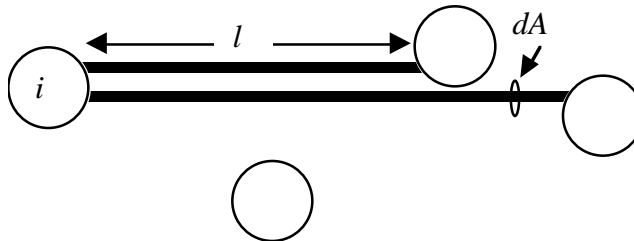
$$\mathbf{v} = \mathbf{v}_0 + \mathbf{a}t = \mathbf{v}_0 + \frac{e\mathbf{E}}{m_e}t. \quad (2)$$

Let's average this either over all of the conduction electrons in the metal or over a large number of scattering events for a given electron. The average velocity equals the drift velocity; the average initial velocity is zero assuming the direction of travel is random after each collision; and the average time between collisions is called the collision time  $\tau$ . Taking the magnitude of the result gives  $v_d = eE\tau / m_e$ , so that Eq. (2) becomes

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_d \frac{t}{\tau}. \quad (3)$$

If we take the magnitude of this equation and note that  $t/\tau$  is of order unity, then we find  $v \equiv v_0$  because the drift speed is negligibly small in comparison (cf. Tipler example 26.1). Thus the average speed of the electrons carrying the current is independent of the applied electric field; for simplicity of notation, I will redefine  $v$  to be this average speed.

Multiplying the collision time by the average speed of these electrons gives the average distance traveled between collisions. This is called the mean free path,  $\lambda = v\tau$ , and it can be related to the scattering cross section  $\sigma$  (often quoted in units of barns = 100 fm<sup>2</sup>) as follows.



Consider the  $i^{\text{th}}$  scattering center (where  $i$  ranges from 1 to  $N$ ). Draw all possible cylindrical

filaments of differentially small cross-sectional area  $dA$  and (variable) length  $l$  which start at this center and extend rightward until they hit the first scattering site in their path. (For clarity, only two representative filaments are shown above. Such filaments represent possible electron trajectories.) Now repeat this process for all  $N$  scatterers. Clearly the total volume  $V$  occupied by these filaments will be that of the entire metal sample (neglecting the small volume taken up by the centers themselves). But averaging the lengths  $l$  of all of these filaments gives the mean free path,

$$\lambda = \frac{\sum_{i=1}^N \int l dA}{N \int dA} = \frac{V}{N\sigma}, \quad (4)$$

where the area integrals were taken over the cross section of a scattering center. Thus  $\lambda = 1/n\sigma$ , where  $n$  is the number density of scatterers. (This derivation is an improvement over Tipler Fig. 27.1 which neglects the possibility that an electron can skim by an atom without colliding with it, as for the lower filament in my diagram.)

Putting together all of the relationships delineated in this section leads to the basic formula

$$\rho = \frac{m_e v n \sigma}{n_e e^2}. \quad (5)$$

Since none of these quantities depends on the electric field, we have thus proved Ohm's law.

## II. Classical Model for Resistivity

Modeling the conduction electrons as a Maxwell-Boltzmann gas implies

$$K_{ave} \equiv \frac{1}{2} m_e v^2 = \frac{3}{2} k_B T \quad \Rightarrow \quad v = \sqrt{\frac{3k_B T}{m_e}}, \quad (6)$$

according to kinetic theory. (We can immediately see this by applying the equipartition theorem: three translational degrees of freedom times the mean energy per degree of freedom.) This works out to be  $1.2 \times 10^5$  m/s at RT, which as claimed above is much larger than typical drift speeds of order 0.1 mm/s.

We further suppose that the electrons scatter off the metal ions like the marbles off the nails in Tipler Fig. 26.8. That is, we put  $n = n_a$  and  $\sigma = \pi R_a^2$  where  $R_a$  is the atomic radius. This latter quantity is ill-defined, but a reasonable approximation might be the Bohr radius,  $a_0$ . (The decreased radii of the orbitals for atomic numbers beyond hydrogen is roughly balanced by the increased number of occupied orbitals.) Our classical prediction for the resistivity thus becomes

$$\rho = \frac{\pi a_0^2}{e^2} \sqrt{3m_e k_B T}. \quad (7)$$

This has two immediate problems: it gives the same result for *all* metals and its temperature dependence is in conflict with the experimentally known linear variation. It does however correctly predict the approximate magnitude of the resistivity of copper at RT.

### III. Quantum Mechanical Model for Resistivity

Electrons actually obey Fermi-Dirac statistics. The most energetic conduction electrons have energies approximately equal to the zero-temperature Fermi energy even at RT, since a typical thermal energy  $k_B T$  is only 25 meV at 293 K and hence there is minimal occupancy of levels higher than this. The Heisenberg uncertainty principle can be used to estimate this energy,

$$\Delta x \Delta p_x \cong h \Rightarrow (\Delta x \Delta y \Delta z)(\Delta p_x \Delta p_y \Delta p_z) \cong h^3. \quad (8)$$

But the volume available per electron is  $\Delta x \Delta y \Delta z = 1/n_e$  and each component of its momentum can range from  $-p_F$  up to  $+p_F$  where  $E_F \equiv p_F^2/2m_e$ . Thus,

$$\frac{1}{n_e} \left( 2\sqrt{2m_e E_F} \right)^3 \cong h^3 \Rightarrow E_F \equiv k_B T_F \cong \frac{h^2 n_e^{2/3}}{8m_e}. \quad (9)$$

This equals about 7 eV for copper (corresponding to a temperature of order  $10^5$  K), which is indeed much larger than 25 meV.

As explained in connection with Tipler Fig. 27.6, while an applied electric field accelerates all of the mobile electrons, the net effect is as though only electrons near the top of the conduction band carry the current. These electrons have the Fermi speed,

$$E_F \equiv \frac{1}{2} m_e v^2 \Rightarrow v = \sqrt{\frac{2k_B T_F}{m_e}}. \quad (10)$$

This is substantially larger than the prediction of Eq. (6) and is very nearly temperature independent provided  $T \ll T_F$ , as is the case for ambient temperatures.

Next we need to re-examine the mean free path. Consider constructing a metal from a set of isolated atoms. As we push the atoms together to build up a solid conductor, the tails of adjacent atomic potential energy functions begin to overlap. This depresses the net potential energy at the saddle point midway between two neighbors until it falls below the energy of the highest occupied level. At that point, the valence electrons in these levels (which split into a closely spaced group of states composing the conduction band) can travel freely from atom to atom without collisions! The wavefunctions of these mobile electrons are called Bloch states and already reflect the periodicity of the underlying metal lattice. In a periodic array, a wave can propagate without attenuation because of coherent constructive interference in the forward direction.

This picture only holds if the lattice has perfect crystalline symmetry. In reality, electrons can still scatter off two sources of imperfections: impurities (such as foreign atoms or vacancies) and thermal vibrations (called phonons) which displace the metal ions from their equilibrium sites. Consequently we calculate the net reciprocal mean free path from Eq. (4) by summing these two contributions,

$$\lambda^{-1} = \sum n\sigma = n_a \sigma_T + n_i \sigma_i \equiv \lambda_T^{-1} + \lambda_i^{-1} \quad (11)$$

where  $n_i$  is the number density of impurities of average scattering cross section  $\sigma_i$ , and  $\sigma_T$  is the effective scattering cross section of the metal atoms due to thermal vibrations. To calculate the latter quantity, model each atom as a mass on a spring oscillating about the lattice position and

apply equipartition to the two translational degrees of freedom of an atom in a plane perpendicular to the direction of motion of an oncoming electron,

$$\frac{1}{2} k_s R_T^2 = E_{\text{vibrational}} = k_B T. \quad (12)$$

Here the amplitude  $R_T$  of the thermal oscillations is related to the cross section by

$$\sigma_T = \pi R_T^2 = \frac{2\pi k_B}{k_s} T. \quad (13)$$

We can estimate the effective spring constant  $k_s$  of the metal ion (which has a charge of  $+e$ ) as follows. When it is displaced a small distance  $r$  from its equilibrium site, the restoring force is largely due to the free electron charge  $-q$  within a sphere of radius  $r$  centered on this site. But on average the volume available to each conduction electron is  $1/n_e \equiv 4\pi R^3/3$ , so that  $q/e = (r/R)^3$ . Hence the restoring force is

$$\frac{eq}{4\pi \epsilon_0 r^2} = \frac{e^2}{4\pi \epsilon_0 R^3} r \equiv k_s r \Rightarrow k_s = \frac{n_e e^2}{3 \epsilon_0}. \quad (14)$$

This works out to be of order 100 N/m for copper, a reasonable value.

Putting everything together and substituting into Eq. (5) gives our final expression for the resistivity,

$$\rho = \frac{h n_e^{1/3}}{2e^2} \left( \frac{6\pi \epsilon_0 k_B}{n_e e^2} T + C_i \sigma_i \right) \quad (15)$$

where  $C_i \equiv n_i/n_a$  is the concentration of impurity atoms. Unlike Eq. (7), this correctly has a linear temperature dependence at high temperatures and levels off to some constant value near absolute zero. Assuming the impurity concentration is small, so that we can neglect the latter term at RT ( $T_0 \equiv 293$  K), this equation gives a resistivity of  $\rho_0 = 1.8 \times 10^{-8} \Omega \cdot \text{m}$  for copper and a temperature coefficient of

$$\alpha \equiv \frac{1}{\rho_0} \frac{d\rho}{dT} = \frac{1}{T_0} = 3.4 \times 10^{-3} / ^\circ\text{C}. \quad (16)$$

These are in excellent agreement (in light of the simplicity of the model) with the experimental values of  $1.7 \times 10^{-8} \Omega \cdot \text{m}$  and  $3.9 \times 10^{-3} / ^\circ\text{C}$ , respectively, from Tipler Table 26.1.