Coriolis Correction to Freefall—C.E. Mungan, Fall 2005

Newton's second law in Earth's frame of reference for the acceleration of a falling rock at position $\mathbf{r}$ is

$$\ddot{\mathbf{r}} = \mathbf{g} + 2\dot{\mathbf{r}} \times \mathbf{\Omega} \tag{1}$$

where $\mathbf{g}$ is Earth's gravitational field including the centrifugal correction and $\mathbf{\Omega}$ is Earth's angular velocity, with magnitude $\Omega = 2\pi / (24 \cdot 3600 \text{ s})$. Suppose the rock is dropped from rest from a height h above the surface, and choose the coordinate system to have its origin on Earth's surface directly below the initial position of the rock. Specifically, let z be straight up (i.e., opposite the direction of $\mathbf{g}$ that a plumb bob hangs at rest), and x and y be due east and north, respectively, perpendicular to the z-axis. The Cartesian components of Eq. (1) then become

$$\begin{aligned} \ddot{x} &= 2\Omega(\dot{y}\cos\theta - \dot{z}\sin\theta) \\ \ddot{y} &= -2\Omega\dot{x}\cos\theta \\ \ddot{z} &= -g + 2\Omega\dot{x}\sin\theta \end{aligned} \tag{2}$$

as in Taylor Eq. (9.53), where θ is the colatitude (the complement of the latitude) at the rock's location. (Actually we should be using $\theta - \alpha$ in place of θ in these equations, where α is the small deflection of $\mathbf{g}$ away from the radial direction toward the equator, i.e., southward in the northern hemisphere. However, since α is no larger than 0.1°, we can neglect this correction.) The problem consists in solving Eq. (2) to find the eastward deflection of the falling rock.

Taylor solves these equations by successive approximations, but as Jordan Kehrer points out, it is not hard to explicitly decouple the x equation and solve it exactly as follows. Integrate the y and z equations once each to get

$$\begin{aligned} \dot{y} &= -2\Omega x\cos\theta + c_1 \\ \dot{z} &= -gt + 2\Omega x\sin\theta + c_2. \end{aligned} \tag{3}$$

The initial conditions are

$$x_0 = y_0 = 0, \quad z_0 = h, \quad \text{and} \quad \dot{x}_0 = \dot{y}_0 = \dot{z}_0 = 0 \tag{4}$$

which implies that $c_1 = c_2 = 0$. We now substitute Eq. (3) into the first equality in Eq. (2) to obtain

$$\ddot{x} = -4\Omega^2 x + 2\Omega g t \sin\theta\,. \tag{5}$$

The complementary solution of the homogeneous equation is a simple sinusoid, while a particular solution $x_\text{p}(t)$ can be found by trying $\ddot{x}_\text{p} = 0$ to get

$$x_\text{p} = \frac{gt\sin\theta}{2\Omega} . \tag{6}$$

Therefore the general solution of Eq. (5) is

$$x = A\cos(2\Omega t) + B\sin(2\Omega t) + \frac{gt\sin\theta}{2\Omega} \tag{7}$$

as the reader is invited to check by explicit substitution into Eq. (5). We again fit to the initial conditions (4) to find

$$A = 0 \quad \text{and} \quad B = -\frac{g\sin\theta}{4\Omega^2} , \tag{8}$$

so that Eq. (7) becomes

$$\boxed{x = \frac{g\sin\theta}{4\Omega^2}\left[2\Omega t - \sin(2\Omega t)\right]}. \tag{9}$$

Noting that $\sin z = z - \frac{1}{6}z^3 + O(z^5)$, we see that Eq. (9) to lowest nonzero order becomes

$$x \cong \tfrac{1}{3}\Omega g t^3 \sin\theta , \tag{10}$$

in agreement with Taylor Eq. (9.57).

If we now substitute Eq. (9) into the second equality in Eq. (2), we find

$$\ddot{y} = g\cos\theta\sin\theta\left[\cos(2\Omega t) - 1\right] \tag{11}$$

which has solution

$$y = -\frac{1}{2}gt^2\cos\theta\sin\theta - \frac{g\cos\theta\sin\theta}{4\Omega^2}\cos(2\Omega t) + c_3 + c_4 t . \tag{12}$$

The initial conditions (4) imply that

$$c_3 = \frac{g\cos\theta\sin\theta}{4\Omega^2} \quad \text{and} \quad c_4 = 0 \tag{13}$$

so that

$$\boxed{y = \frac{g\cos\theta\sin\theta}{4\Omega^2}\left[1 - \tfrac{1}{2}(2\Omega t)^2 - \cos(2\Omega t)\right]}. \tag{14}$$

Noting that $\cos z = 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 + O(z^6)$, Eq. (14) to lowest nonzero order becomes

$$y \cong -\tfrac{1}{6}\Omega^2 g t^4 \cos\theta\sin\theta . \tag{15}$$

While this correctly has a southerly direction, and has the right functional dependence on the various parameters, the prefactor 1/6 is wrong, as explained in problem 10-13 of Thornton &

Marion (5th edition). Namely, there are two other corrections with the same functional dependence as Eq. (15), due to the variation in the inertial gravitational field and the centrifugal acceleration with height. Also see the footnote on page 401 of Thornton & Marion for a reference to the numerical value of this southerly deflection and attempts to measure it experimentally.