Cosines of Common Angles—C.E. Mungan, Fall 2008

There is a pattern to the cosines of the commonly memorized angles, as nicely diagrammed in the unit circle on the middle of the Wikipedia page about the unit circle:

ϕ	$\cos\phi$
30°	$\dfrac{\sqrt{3}}{2}$
45°	$\dfrac{\sqrt{2}}{2}$
60°	$\dfrac{\sqrt{1}}{2}$

We can improve on the trend by adding in two more angles:

ϕ	$\cos\phi$
0°	$\dfrac{\sqrt{4}}{2}$
30°	$\dfrac{\sqrt{3}}{2}$
45°	$\dfrac{\sqrt{2}}{2}$
60°	$\dfrac{\sqrt{1}}{2}$
90°	$\dfrac{\sqrt{0}}{2}$

This looks nice, except the angles don't divide up the first quadrant evenly. Let's add in 15° and 75°, whose cosines are easily worked out using the half-angle formulae:

ϕ	$\cos\phi$
0°	$\frac{\sqrt{2+\sqrt{4}}}{2}$
15°	$\frac{\sqrt{2+\sqrt{3}}}{2}$
30°	$\frac{\sqrt{2+\sqrt{1}}}{2}$
45°	$\frac{\sqrt{2+\sqrt{0}}}{2}$
60°	$\frac{\sqrt{2-\sqrt{1}}}{2}$
75°	$\frac{\sqrt{2-\sqrt{3}}}{2}$
90°	$\frac{\sqrt{2-\sqrt{4}}}{2}$

Great, the first quadrant is uniformly divided into 15° segments. But the symmetry of the cosine values is now imperfect: What happened to

$$\frac{\sqrt{2+\sqrt{2}}}{2} \quad \text{and} \quad \frac{\sqrt{2-\sqrt{2}}}{2}\ ?$$

As you can easily check, they're equal to cosine of 22.5° and 67.5°! It's no coincidence that the second table in this note (which had a uniform pattern to the cosine values) was symmetric about 45°, and now we're finding a symmetry about half of that angle (i.e., 22.5°) when we again write down a uniform pattern of cosine values. To see the trend take the inverse cosine of the following angle in degrees and don't be too surprised at what you get:

$$\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}.$$