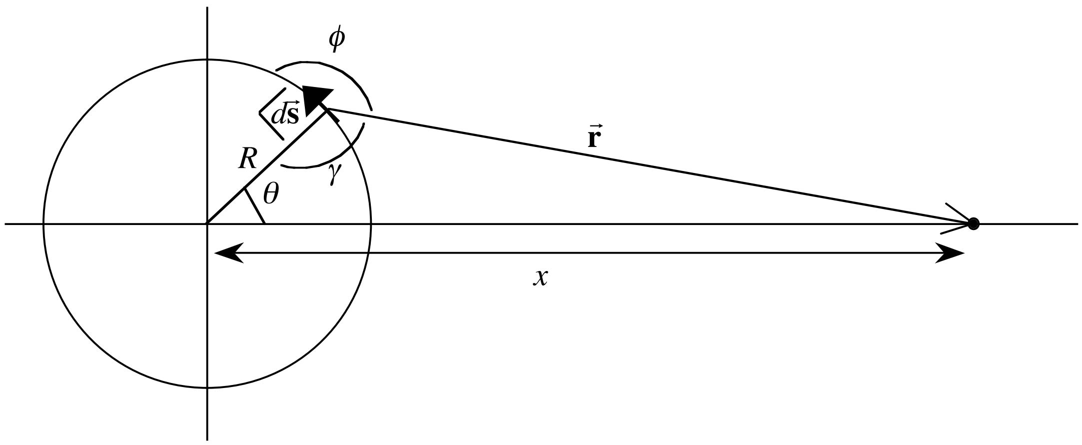
Magnetic Field of a Current Loop in the Plane of the Loop—C.E. Mungan, Spring 2004

The goal of this exercise is to compute the magnetic field at distance x from the center of a thin wire loop of radius R carrying counter-clockwise current I, as sketched below.



Letting the positive direction be out of the page, the Biot-Savart law implies that the magnetic field is

$$B = -\frac{\mu_0 I}{4\pi}\int_0^{2\pi}\frac{R\,d\theta\sin\phi}{r^2}. \tag{1}$$

We see from the diagram that

$$\phi + \gamma = 270^\circ \quad \Rightarrow \quad \sin\phi = -\cos\gamma \tag{2}$$

and

$$x^2 = R^2 + r^2 - 2Rr\cos\gamma \tag{3}$$

from the law of cosines. Combining Eqs. (2) and (3) leads to

$$\sin\phi = \frac{x^2 - R^2 - r^2}{2Rr}. \tag{4}$$

Another application of the law of cosines gives

$$r^2 = R^2 + x^2 - 2Rx\cos\theta. \tag{5}$$

Substituting Eqs. (4) and (5) into (1) and rearranging implies

$$\boxed{B = -\frac{\mu_0 I R^2}{4\pi x^3}\int_0^{2\pi}\left(\frac{x}{R}\cos\theta - 1\right)\left(1 + \frac{R^2}{x^2} - \frac{2R}{x}\cos\theta\right)^{-3/2} d\theta}. \tag{6}$$

We can check that Eq. (6) gives the right answer in two limits. First, at the center of the loop, $x = 0$ so that

$$B = \frac{\mu_0 I}{4\pi R}\int_0^{2\pi} d\theta = \frac{\mu_0 I}{2R} \equiv B_0 \tag{7}$$

as is well known. Second, for large *x*, we can approximate the second term in parentheses using a first-order binomial expansion in R/x, to obtain

$$\begin{aligned} B &\cong -\frac{\mu_0 I R^2}{4\pi x^3}\int_0^{2\pi}\left(\frac{x}{R}\cos\theta - 1\right)\left(1 + \frac{3R}{x}\cos\theta\right)d\theta \\ &= -\frac{\mu_0 I R^2}{4\pi x^3}\int_0^{2\pi}\left(\frac{x}{R}\cos\theta - 1 - \frac{3R}{x}\cos\theta + 3\cos^2\theta\right)d\theta \\ &= -\frac{\mu_0 I R^2}{4\pi x^3}(2\pi)\left(0 - 1 - 0 + \tfrac{3}{2}\right) \end{aligned} \tag{8}$$

or

$$\boxed{\vec{\mathbf{B}} \cong -\frac{\mu_0 \vec{\mathbf{m}}}{4\pi x^3}} \tag{9}$$

where $\vec{\mathbf{m}}$ is the magnetic dipole moment of the loop. This has the familiar inverse-cube dependence on distance.

For intermediate values of *x*, Eq. (6) cannot be solved analytically. The following command can be used in Mathematica to numerically integrate it,

```
f[d_]:=NIntegrate[{(d*Cos[x]-1)*(1+1/d^2-2/d*Cos[x])^(-1.5)},{x,0,2*Pi}]/(2*Pi*d^3)
```

where $d \equiv x/R$ is the normalized distance and $f(d) \equiv B/B_0$ is the normalized magnetic field. Then the command

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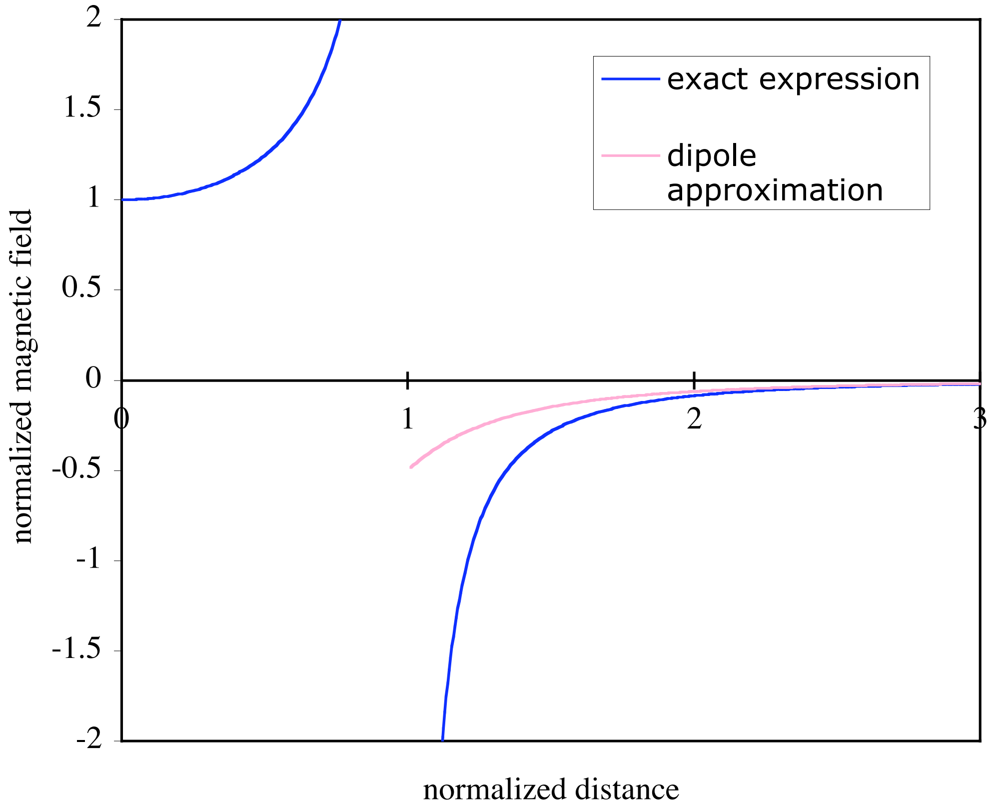
TableForm[N[Table[{d,f[d]},{d,0,3,0.01}]]]
```

can be used to generate a 2-column table of values which can be copied and pasted into Excel. This is plotted at the top of the next page. Note that Eq. (6) reduces to

$$B = \frac{\mu_0 I}{8\pi R}\int_0^{\pi}\csc\alpha \, d\alpha \tag{10}$$

at $x = R$, which diverges as one might expect when one hits the wire.



The other curve is a plot of Eq. (9), which consequently is seen to be a reasonable approximation for $x > 2R$.