Fastest Descent along a Ramp and Horizontal Track—C.E. Mungan, Fall 2016

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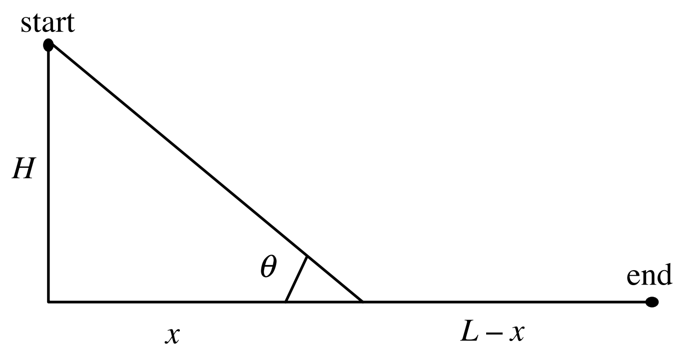
Starting at rest at height H above the ground, an object slides frictionlessly down an inclined plane and then along a horizontal track, such that it ends a horizontal distance L from where it started, as sketched in the following figure. Find the value of the incline angle θ that minimizes the total transit time t. Also show that there is some minimum value of L for this solution to exist, as otherwise the descent is optimal along a ramp extending directly from the starting to the ending point (which is the path of shortest distance between those two points).



The descent time along the ramp is the distance $(H^2 + x^2)^{1/2}$ along it divided by the average speed $\upsilon_f / 2$ along it, where the speed at the bottom is $\upsilon_f = (2gH)^{1/2}$ from energy conservation. Next the transit time along the horizontal track is its length $L - x$ divided by the constant speed υ_f on it. Thus the total travel time is

$$t = \frac{2\sqrt{H^2 + x^2} + L - x}{\sqrt{2gH}} . \tag{1}$$

Recast this in dimensionless form by defining $T \equiv (2g/H)^{1/2} t$ and $X \equiv x/H$ to get

$$T = 2\sqrt{1 + X^2} - X + L/H . \tag{2}$$

Thus the total time is minimized by finding the value of X that minimizes

$$Y = 2\sqrt{1 + X^2} - X . \tag{3}$$

Add X to both sides, square that equation, and then rearrange the result to get

$$3X^2 - 2YX + (4 - Y^2) = 0 . \tag{4}$$

This quadratic equation in X is solved using the quadratic formula to obtain

$$X = \frac{Y \pm 2\sqrt{Y^2 - 3}}{3} . \tag{5}$$

Physically we require both X and Y to be positive. Starting from large positive values of Y, we have to choose the plus sign in Eq. (5) to get a positive value of X. Now progressively decrease the value of Y. Once it becomes smaller than 2, we can choose either sign in Eq. (5) and get two positive values[1] of X. As we continue to decrease the value of Y still more, we see from Eq. (5) that these two values of X converge toward one common value when $Y = 3^{1/2}$. We cannot decrease Y any more than that, because then Eq. (5) has no real solutions for X. Therefore, the smallest value of Y that gives a positive real solution for X is

$$Y_{\text{min}} = \sqrt{3}\,. \tag{6}$$

If we substitute that value of Y back into Eq. (5), we see that it occurs at a value of X equal to

$$X_{\text{min}} = 1/\sqrt{3}\,. \tag{7}$$

Therefore the optimal ramp angle is

$$\theta_{\text{min}} = \tan^{-1}\sqrt{3} = 60° \tag{8}$$

provided that $L > x_{\text{min}} = H/\sqrt{3} \approx 0.58H$.

[1]Note that $X = 0$ when $Y = 2$ if we choose the minus sign in Eq. (5); this solution corresponds to a vertical ramp with $\theta = 90°$. On the other hand, if $Y = 2$ and we choose the positive sign in Eq. (5), then $X = 4/3$ gives the same total travel time $T = 2 + L/H$ that $X = 0$ does. This $X = 4/3$ solution corresponds to a ramp angle of $\theta = \tan^{-1} 0.75 \approx 37°$ (provided $L > 4H/3$).