A Roll of the Dice—C.E. Mungan, Spring 2023

There are $N \to \infty$ gamblers in a casino. Each gambler is given a fair six-sided die. They each roll their die up to M times and write down on a card how many rolls n it takes until a 4 (or any other number between 1 and 6 that you like) first appears on the die. They then hold their cards up in the air. If a 4 does not appear within M rolls, they do not write down a value nor do they hold their card up in the air. You look around the room and average the values you see on the cards. What average value n_{avg} do you obtain?

A 4 first appears on the n-th roll if the first $n-1$ rolls are not a 4, each of which has probability 5/6 of occurring, and a 4 occurs on the n-th roll, which has probability 1/6. Thus the unnormalized probability a gambler will report a value of n is

$$p(n) = \left(\tfrac{5}{6}\right)^{n-1}\left(\tfrac{1}{6}\right). \tag{1}$$

Multiply this probability by the number of rolls n, sum over all possible values of n from 1 to M, and normalize the result to obtain

$$n_{\text{avg}} = \frac{\sum_{n=1}^{M} np(n)}{\sum_{n=1}^{M} p(n)}. \tag{2}$$

Each die has $x = 6$ sides, so that the sum in the denominator of Eq. (2) can be written as

$$\sum_{n=1}^{M} p(n) = \sum_{n=1}^{M}\left(1-\frac{1}{x}\right)^{n-1}\frac{1}{x} = \frac{1}{x}\sum_{m=0}^{M-1} q^m = \frac{1}{x}\frac{1-q^M}{1-q} = 1-\left(1-\frac{1}{x}\right)^M = 1-\left(\frac{5}{6}\right)^M \tag{3}$$

making temporary use of $q \equiv 1-1/x$ and $m \equiv n-1$. Similarly using $y = 1/6$, the sum in the numerator of Eq. (2) becomes

$$\begin{aligned}\sum_{n=1}^{M} np(n) &= -\sum_{n=1}^{M} n(1-y)^{n-1}(1-y-1) \\ &= -\sum_{n=1}^{M}(n+1)(1-y)^n + \sum_{n=1}^{M}(1-y)^n + \sum_{n=1}^{M} n(1-y)^{n-1} \\ &= \frac{d}{dy}\sum_{n=1}^{M}(1-y)^{n+1} + \sum_{n=1}^{M}(1-y)^n - \frac{d}{dy}\sum_{n=1}^{M}(1-y)^n.\end{aligned} \tag{4}$$

But

$$\sum_{n=1}^{M}(1-y)^n = \sum_{n=0}^{M}(1-y)^n - 1 = \frac{1-(1-y)^{M+1}}{y} - 1 \tag{5}$$

so that

$$-\frac{d}{dy}\sum_{n=1}^{M}(1-y)^n = -\frac{(M+1)(1-y)^M}{y} + \frac{1-(1-y)^{M+1}}{y^2} \tag{6}$$

and

$$\begin{aligned}\frac{d}{dy}\sum_{n=1}^{M}(1-y)^{n+1} &= \frac{d}{dy}\frac{(1-y)^2-(1-y)^{M+2}}{y} \\ &= \frac{-2(1-y)+(M+2)(1-y)^{M+1}}{y} - \frac{(1-y)^2-(1-y)^{M+2}}{y^2}\end{aligned} \tag{7}$$

so that finally

$$n_{\text{avg}} = \boxed{6-\frac{M}{1.2^M-1}} \tag{8}$$

after substituting Eqs. (5) to (7) into (4) and putting $y = 1/6$, and then substituting Eqs. (3) and (4) into (2). This result correctly equals 1 when $M = 1$ (because the only number held up on the cards is 1) and it asymptotically approaches 6 (using l'Hôpital's rule) as $M \to \infty$. The significance of this asymptotic value is that one expects it to take six rolls of a die on average for it to first show a value of 4. That is perhaps not too surprising, because there are six possible values a die can show. Equation (8) is plotted below and is within 1% of its limiting value of 6 for all $M \geq 35$.