Quick Reference to Elementary Methods for Solving Differential Equations
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First-Order Linear $\quad y' + P(x)y = Q(x)$

The homogeneous equation is separable, giving $y = Ae^{-I}$ where $\boxed{I \equiv \int P dx} \Rightarrow ye^{I} = \text{constant}$. Now observe in the inhomogeneous case that $\frac{d}{dx}\left(ye^{I}\right) = e^{I}\left(y' + Py\right) = Qe^{I}$, so that the general solution is $\boxed{ye^{I} = \int Qe^{I} dx + c}$.

Second-Order Linear Homogeneous with Constant Coefficients

$(D-a)(D-b)y = 0$ with $a \neq b$ has solution $\boxed{y = c_1 e^{ax} + c_2 e^{bx}}$ by inspection, while $(D-a)(D-a)y = 0$ has solution $\boxed{y = c_1 e^{ax} + c_2 x e^{ax}}$ where the second solution comes from solving the first-order equation $(D-a)y = e^{ax}$.

Second-Order Linear Inhomogeneous with Constant Coefficients

The general solution is $\boxed{y = y_c + y_p}$, where y_p is any particular solution of the inhomogeneous equation and y_c is the general solution of the homogeneous equation. A particular solution of $(D-a)(D-b)y = e^{cx}P_n(x)$, where P_n is any polynomial of degree n, is

$$\boxed{y_p = e^{cx} Q_n(x) \times \begin{cases} 1 & \text{if} \quad c \neq a, b \\ x & c = a \text{ or } c = b \text{ but } a \neq b \\ x^2 & c = a = b \end{cases}}$$

where c may be real, complex, or zero. Here Q_n is a polynomial of degree n whose coefficients are to be determined by substituting into the differential equation and equating coefficients of like powers of x.

Other Second-Order Equations

- If there is no y in the equation, then let $\boxed{z \equiv y'}$ to get a first-order equation for $z(x)$.
- If there is no x in the equation, then again let $\boxed{z \equiv y'} \Rightarrow y'' = z\frac{dz}{dy}$ to get a first-order equation for $z(y)$.
- The solutions of the homogeneous Euler equation $a_2 x^2 y'' + a_1 x y' + a_0 y = 0$ are obviously of the form $\boxed{y = x^n}$; if substitution gives a quadratic equation for n with a double root, then the second solution is $\boxed{y = x^n \ln x}$, as can be proven from the following.
- If one solution of the homogeneous equation $y'' + p(x)y' + q(x)y = 0$ is $y = u(x)$, then the second solution is $\boxed{y = u\int u^{-2} e^{-J} dx}$ with $\boxed{J \equiv \int p dx}$. More generally, if u is a solution of an inhomogeneous equation, then $u\upsilon$ is a second solution and is found by substituting into the differential equation to get a first-order equation for υ.