Direct Harmonic Balance—C.E. Mungan, Fall 2014

Gottlieb [1] has argued that the most direct harmonic balance method consists in finding the first Fourier coefficient of the restoring force. That is calculated as follows. Newton's second law for the displacement x as a function of time t is

$$\frac{d^2x}{dt^2} = -f(x) \tag{1}$$

where $f(x)$ is the negative of the force per unit mass. The harmonic balance method consists in assuming a periodic solution with amplitude A and angular frequency ω,

$$x(t) = A\cos\omega t \tag{2}$$

by appropriate choice of zero time. Substituting Eq. (2) into (1) gives

$$A\omega^2 \cos\theta = f(A\cos\theta) \tag{3}$$

where the phase is $\theta = \omega t$. Now imagine expanding the right-hand side in a Fourier cosine series in θ. Pick off the first coefficient by taking the inner product of the force with $\cos\theta$ and identifying that with the first coefficient on the left-hand side to get

$$A\omega^2 = \frac{1}{\pi}\int_0^{2\pi} f(A\cos\theta)\cos\theta\, d\theta\,. \tag{4}$$

Owing to the periodicity of the restoring force, the integral can be calculated over only the first quadrant of the phase angle to end up with

$$\omega = \sqrt{\frac{4}{A\pi}\int_0^{\pi/2} f(A\cos\theta)\cos\theta\, d\theta} \tag{5}$$

as the estimated oscillational frequency. Compare this approximation to the five cases treated in the double harmonic balance (HB) paper [2] and then consider a cube root oscillator [3].

Case 1: The simple harmonic oscillator

The exact solution and the double HB method both predict

$$\omega_{\text{exact}} = \omega_0 \quad \text{for } f(x) = \omega_0^2 x\,, \tag{6}$$

as does Eq. (5),

$$\omega = \sqrt{\frac{4}{A\pi}\int_0^{\pi/2} \omega_0^2 A\cos\theta\cos\theta\, d\theta}\,. \tag{7}$$

Case 2: Motion down two intersecting planes

The exact solution and the double HB method both predict

$$\omega_{\text{exact}} = \frac{\pi}{2\sqrt{2}}\omega_0 \approx 1.111\omega_0 \quad \text{for } f(x) = A\omega_0^2 \,\text{sgn}(x)\,, \tag{8}$$

whereas Eq. (5) becomes

$$\omega = \sqrt{\frac{4}{A\pi}\int_0^{\pi/2} A\omega_0^2 \cos\theta \, d\theta} = \frac{2}{\sqrt{\pi}}\omega_0 \approx 1.128\omega_0 \tag{9}$$

which is 1.6% too big.

Case 3: Electric force inversely proportional to distance

The exact solution is

$$\omega_{\text{exact}} = \sqrt{\frac{\pi}{2}}\omega_0 \approx 1.253\omega_0 \quad \text{for } f(x) = A^2\omega_0^2 \big/ x\,, \tag{10}$$

whereas the double HB method gives a prediction in terms of the Catalan constant G as

$$\omega = \sqrt{2G}\omega_0 \approx 1.353\omega_0 \tag{11}$$

which is 8% too big. In contrast, Eq. (5) predicts

$$\omega = \sqrt{\frac{4}{A\pi}\int_0^{\pi/2} \frac{A^2\omega_0^2}{A\cos\theta}\cos\theta \, d\theta} = \sqrt{2}\omega_0 \approx 1.414\omega_0 \tag{12}$$

that is 13% too large.

Case 4: The large-amplitude simple pendulum

The exact solution with angular amplitude $A = \pi/2$ is

$$\omega_{\text{exact}} = \frac{2\pi^{3/2}}{\Gamma^2(1/4)}\omega_0 \approx 0.8472\omega_0 \quad \text{for } f(x) = \omega_0^2 \sin(x)\,. \tag{13}$$

The double HB method does not give an analytical result, but a numerical approximation is

$$\omega \approx 0.8438\omega_0 \tag{14}$$

which is 0.4% too small. Equation (5) predicts

$$\omega = \sqrt{\frac{8}{\pi^2}\int_0^{\pi/2} \omega_0^2 \sin\left(\frac{\pi}{2}\cos\theta\right)\cos\theta \, d\theta} = 2\sqrt{\frac{J_1(\pi/2)}{\pi}}\omega_0 \approx 0.8495\omega_0 \tag{15}$$

that is 0.3% too big.

Case 5: The Duffing oscillator with zero linear term

The exact solution is

$$\omega_{\text{exact}} = \frac{2\pi^{3/2}}{\Gamma^2(1/4)}\omega_0 \approx 0.8472\omega_0 \quad \text{for } f(x) = \omega_0^2 x^3 \big/ A^2 \ , \tag{16}$$

whereas Eq. (5) predicts

$$\omega = \sqrt{\frac{4}{A\pi}\int_0^{\pi/2} \omega_0^2 \frac{A^3\cos^3\theta}{A^2}\cos\theta\, d\theta} = \frac{\sqrt{3}}{2}\omega_0 \approx 0.8660\omega_0 \tag{17}$$

which is 2% too big. The double HB method to first order makes the same prediction.

Case 6: The cube root oscillator

The exact solution for

$$f(x) = \omega_0^2 A^{2/3} x^{1/3} \tag{18}$$

is found using Eq. (19) in Ref. [2] to be

$$\omega_{\text{exact}} = \frac{\pi}{2}\sqrt{\frac{3}{2}}\left[\int_0^1 \frac{du}{\sqrt{1-u^{4/3}}}\right]^{-1}\omega_0 = \frac{\sqrt{3\pi/8}\,\Gamma(5/4)}{\Gamma(7/4)}\omega_0 \approx 1.07045\omega_0 \,, \tag{19}$$

whereas Eq. (5) predicts

$$\omega = \sqrt{\frac{4}{A\pi}\int_0^{\pi/2} \omega_0^2 A^{2/3} A^{1/3}\cos^{1/3}\theta\cos\theta\, d\theta} = \frac{\omega_0}{\pi^{1/4}}\sqrt{\frac{3\Gamma(7/6)}{\Gamma(2/3)}} \approx 1.07685\omega_0 \tag{20}$$

which is only 0.6% too big. The double HB method predicts

$$\omega = 2\omega_0\sqrt{\sum_{n=1}^{n_{\max}}\int_0^1 u^{1/3}\sin(n\pi u)\,du \sum_{k=0}^{k_{\max}} \frac{(-1)^k J_{2k+1}(n\pi)}{(2k+1)^2}} \tag{21}$$

but no analytic solution exists for $n_{\max} = k_{\max} = \infty$. However, the sum over *k* falls off very rapidly with increasing *k*, so it suffices to set $k_{\max} = 20$. Equation (21) is then evaluated numerically in Mathematica for various values of $n_{\max}$ to obtain the following table.

$n_{\max}$	ω / ω_0
10	0.9199
100	1.0196
1000	1.05172
10 000	1.06184

The results are converging toward a value near the exact solution in Eq. (19) but it would take a significant additional amount of computer time to determine how close it ends up getting. The double HB method is most useful when either an analytic solution exists or one can numerically approximate it with only a few terms.

[1] H.P.W. Gottlieb, “Frequencies of oscillators with fractional-power non-linearities,” J. Sound Vib. **261**, 557 (2003).
[2] T.C. Lipscombe and C.E. Mungan, “Double Fourier harmonic balance method for nonlinear oscillators by means of Bessel series,” Int. J. Math. Eng. Sci. **3**, Issue 3, ID 4 (2014).
[3] R.E. Mickens, “Oscillations in an $x^{4/3}$ potential,” J. Sound Vib. **246**, 375 (2001).