Perfectly Inelastic Collision Between a Disk and a Stick—C.E. Mungan, Summer 2012

Example 11.9 in the 7th edition of Serway & Jewett's *Physics for Scientists & Engineers* concerns a disk which strikes the end of a uniform stick perpendicularly. We are given the mass of the disk $m_\text{d} = 2$ kg, its initial speed $\upsilon_\text{di} = 3$ m/s, the mass of the stick $m_\text{s} = 1$ kg, and its length $L = 4$ m. Therefore, the moment of inertia of the stick about its geometric center is $I = m_\text{s}L^2 / 12 = 4/3 \text{ kg}\cdot\text{m}^2$. In the main part of the problem, the collision is assumed to be elastic. On page 325 we are then asked, "What if the collision between the disk and the stick is perfectly inelastic?"

The proffered solution assumes that the disk adheres to the stick on colliding, from which it is calculated that the center of mass (CM) of the stick-disk system is located $y_\text{CM} = 4/3$ m above the midpoint of the stick and has a final translational speed of $\upsilon_\text{CM} = 2$ m/s. The moment of inertia of the stick alone about this new CM is calculated from the parallel-axis theorem to be $I_\text{s} = I + m_\text{s} y_\text{CM}^2 = 28/9 \text{ kg}\cdot\text{m}^2$. The stick and disk subsequently rotate with an angular speed of $\omega = -1$ rad/s, where the minus sign indicates the rotation is clockwise, as expected intuitively.

The problem ends by stating that kinetic energy is lost, since the collision is inelastic. Let's crunch the numbers. The initial kinetic energy of the system is just translational KE of the disk, $K_\text{i} = m_\text{d}\upsilon_\text{di}^2 / 2 = 9$ J. The final KE is the sum of the translational KE of the CM and rotational KE of each object about the new CM, $K_\text{f} = (m_\text{d} + m_\text{s})\upsilon_\text{CM}^2 / 2 + [I_\text{s} + m_\text{d}(\Delta y)]\omega^2 / 2 = 8$ J, where $\Delta y = L/2 - y_\text{CM} = 2/3$ m is the distance from the new CM to the disk on the end of the stick. Thus the lost kinetic energy is $K_\text{lost} \equiv K_\text{i} - K_\text{f} = 1$ J.

However, an alternative solution defines a "perfectly inelastic collision" to be one in which the normal component of the relative velocity between the contact points on the two objects is zero immediately after the collision *without assuming that the two objects adhere together*. Specifically suppose that the collision in the present extension of the problem occurs in such a manner that the disk and the top end of the stick immediately after the collision move to the right with the same speed υ_df, but the disk and stick do NOT adhere together and the stick subsequently rotates away from contact with the disk.

Let's determine what happens in this case. Equations (1) and (2) remain as in the main solution of the problem (prior to the "What If?"), namely

$$m_\text{d}\upsilon_\text{di} = m_\text{d}\upsilon_\text{df} + m_\text{s}\upsilon_\text{s} \tag{1}$$

from conservation of the system linear momentum, where υ_s is the final speed of the center of the stick. Similarly, conservation of system angular momentum gives

$$-m_\text{d}\upsilon_\text{di}(L/2) = -m_\text{d}\upsilon_\text{df}(L/2) + I\omega \tag{2}$$

about the center of the stick. The third equation in the problem assumes that the system kinetic energy is the same before and after the collision; that is no longer true when the collision is inelastic. But assuming the disk and top end of the stick have the same speed immediately after the collision, we instead have the relation

$$v_{\mathrm{df}} = v_{\mathrm{s}} - \omega(L/2)\,. \tag{3}$$

Simultaneous solution of Eqs. (1) to (3) gives $v_{\mathrm{df}} = 8/3$ m/s, $v_{\mathrm{s}} = 2/3$ m/s, and $\omega = -1$ rad/s. Note that we obtained the same final angular speed ω as we did when the stick and disk adhered. Not only that, but the final CM speed is also the same,

$$v_{\mathrm{CM}} = (m_{\mathrm{d}} v_{\mathrm{df}} + m_{\mathrm{s}} v_{\mathrm{s}})/(m_{\mathrm{d}} + m_{\mathrm{s}}) = 2 \text{ m/s}\,.$$

Not surprisingly, since these quantities are all unchanged whether the disk adheres or not, the final KE of the system is also unchanged, even though we compute each contribution to the total differently, $K_{\mathrm{f}} = m_{\mathrm{d}} v_{\mathrm{df}}^2/2 + m_{\mathrm{s}} v_{\mathrm{s}}^2/2 + I\omega^2/2 = 8$ J.

In this case, we therefore found that the book's solution and my alternative solution give the same results. However, as I discuss in EJP **28**, 563 (2007), where that hyperlink is to a corrected version of the paper on my website, if the disk is initially traveling horizontally but strikes the end of the stick at an angle (relative to the axis of the stick) other than 90˚ (or 0˚), these two solutions would no longer give the same results! The disk would then have a component of its initial motion along the axis of the stick. If the disk does not adhere to the stick and there is no friction between them, that component of the disk's motion would remain unaltered after the collision. On the other hand, if the two adhere together, a tangential "bonding" (or "frictional") force will need to absorb some of that component of the disk's initial velocity, so that the disk follows the stick's rotation. Consequently the system will lose more KE than it would in the absence of adhesion. One might therefore think the adhering case is a better candidate as a definition of "perfectly inelastic" in the sense of maximum loss of KE. However, the nonadhering case is the preferred definition by mechanical engineers who study collision problems for the simple reason that they wish to define a coefficient of restitution *e* that smoothly increases from 0 to 1. No adhesion occurs of course when *e* is say 0.01; thus it makes sense to assume none occurs when *e* becomes exactly 0.