The Classical Doppler Effect—C.E. Mungan, Spring 1999

Most introductory textbooks discuss two special cases of the classical Doppler effect—moving source or moving observer. In fact, it may be more helpful to consider three special cases rather than two—moving source (*s*), moving medium (*m*), or moving observer (*o*)—with the other two objects stationary in each case. These three cases can be related to the three primary parameters of a traveling wave—its wavelength λ, frequency f, and speed υ. In fact, each of these three special cases involves exactly one of these three parameters being the same for both source and observer, so there is a nice one-to-one association between special cases and wave parameters. Let me remind you of how it goes, restricting all motions to one dimension. I denote the values of the parameters in the observer's frame of reference by a prime, and those in the source's frame of reference by the lack of a prime. Equivalently, the unprimed quantities are what would be measured by the observer if the source, medium, and observer were all in the same frame of reference (such as if they were all at rest).

<u>Special Case I: Moving Source (with Stationary Medium and Observer)</u>

Here $\upsilon' = \upsilon$ because the medium is at rest relative to the observer. The speed of a wave is determined purely by the medium properties (relative to the observer), quite independent of what the source may be doing.

Consider the case where the source moves toward the observer. In this case, the wavefronts get crowded together, so that we expect $\lambda' < \lambda$. In fact, it is not hard to see that the fractional decrease in the wavelength is equal to the relative source speed, where we define all relative speeds by comparison to the proper (unprimed) sound speed. Thus,

$$\frac{\lambda - \lambda'}{\lambda} = \frac{\upsilon_s}{\upsilon}.$$

Finally, the frequency must increase by exactly the same factor as the wavelength decreased, in order to ensure that $\upsilon' = \upsilon \Rightarrow f'\lambda' = f\lambda$.

Putting everything together, we thus have

$$\boxed{\begin{aligned} &\lambda' = \lambda(1 - \upsilon_s / \upsilon) \\ &f' = f\left(\frac{1}{1 - \upsilon_s / \upsilon}\right) \\ &\upsilon' = \upsilon \end{aligned}} \qquad \text{SOURCE MOVING TOWARD OBSERVER}$$

As a check, note that the observed wavelength and frequency fall to zero and infinity, respectively, if the source travels at mach 1, because the wavefronts are then emitted one right on top of the next. If the source is instead moving away from the observer, one flips the two minus signs to plus signs or you can let υ_s be negative; the same principle holds for all the other cases discussed below.

<u>Special Case II: Moving Observer (with Stationary Source and Medium)</u>

Here $\lambda' = \lambda$ because the medium is at rest relative to the source. Absent special relativistic effects, lengths are frame-invariant quantities.

Consider the case where the observer moves toward the source. In this case, the observer is rushing head-long into the wavefronts, so that we expect $\upsilon' > \upsilon$. In fact, the wave speed is simply increased by the observer speed, as we can see by jumping into the observer's frame of reference. Thus,

$$\upsilon' = \upsilon + \upsilon_o = \upsilon(1 + \upsilon_o / \upsilon).$$

Finally, the frequency must increase by exactly the same factor as the wave speed increased, in order to ensure that $\lambda' = \lambda \Rightarrow \upsilon' / f' = \upsilon / f$.

Putting everything together, we thus have

$$\boxed{\begin{aligned} &\lambda' = \lambda \\ &f' = f(1 + \upsilon_o / \upsilon) \\ &\upsilon' = \upsilon + \upsilon_o \end{aligned}}$$

OBSERVER MOVING TOWARD SOURCE

As a check, note that the observed frequency doubles if the observer travels at mach 1, because the time delay between wavefront encounters is then halved; this is a far smaller frequency shift than the infinite shift found in case I when the source travels at mach 1—the asymmetry arises from the fact that in the present case there is an on-coming wind in the observer's frame of reference which tends to blow the wavefronts apart.

<u>Special Case III: Moving Medium (with Stationary Source and Observer)</u>

Here $f' = f$ because the observer is at rest relative to the source. The frequency of a wave is determined purely by the source properties (relative to the observer), quite independent of what the medium may be doing.

Consider the case where the medium moves from the source toward the observer. In this case, the wavefronts are being blown to the observer, so that we expect $\upsilon' > \upsilon$. In fact, the wave speed is simply increased by the speed of the medium, as we can see by jumping into the medium's frame of reference. Thus,

$$\upsilon' = \upsilon + \upsilon_m = \upsilon(1 + \upsilon_m / \upsilon).$$

Finally, the wavelength must increase by exactly the same factor as the wave speed increased, in order to ensure that $f' = f \Rightarrow \upsilon' / \lambda' = \upsilon / \lambda$.

We thus have

$$\boxed{\begin{aligned} &\lambda' = \lambda(1 + \upsilon_m / \upsilon) \\ &f' = f \\ &\upsilon' = \upsilon + \upsilon_m \end{aligned}}$$

MEDIUM MOVING TOWARD OBSERVER

As a check, imagine a receiver moving along with the medium which intercepts what the source emits, resulting in a downshifted frequency of f'', and then immediately rebroadcasts it to the observer, who receives it with an upshifted frequency of f'. Now $f'' = f/(1 + \upsilon_m / \upsilon)$ from case I, because in the interceptor's frame of reference the medium is at rest and the source is moving away from it at speed υ_m. But $f' = f''(1 + \upsilon_m / \upsilon)$ from case II, since in the interceptor's frame of reference the medium is at rest and the observer is moving toward it at at speed υ_m. Substituting the first expression into the second proves that $f' = f$ so that there is no Doppler shift.

General Case: Source, Medium, and Observer all Moving

We add together the fractional changes in wavelength and velocity for each of the three cases above to get the general result for the observed wavelength and velocity, respectively, and then find the primed frequency using $f' = \upsilon'/\lambda'$. This expression for υ' is simply the relative velocity of sound in the observer's frame of reference. On the other hand, the expression for λ' can be verified as follows. Suppose the source emits a wavefront; in one source period that wavefront travels a distance (in a frame at absolute rest) of $(\upsilon + \upsilon_m)/f$, while the source travels υ_s / f, and the observed wavelength will be equal to the difference between these two distances since that represents the spacing between wavefronts (regardless of whether the observer is moving or not). Thus,

$$\boxed{\begin{aligned} &\lambda' = \lambda(1 + \upsilon_m/\upsilon - \upsilon_s/\upsilon) \\ &f' = f\frac{1 + \upsilon_m/\upsilon + \upsilon_o/\upsilon}{1 + \upsilon_m/\upsilon - \upsilon_s/\upsilon} \\ &\upsilon' = \upsilon + \upsilon_m + \upsilon_o \end{aligned}} \qquad \text{MOVING TOWARD}$$

where "moving toward" means that the source and medium are moving toward the observer, and the observer is moving toward the source, so that one gets a frequency upshift for any values of υ_s, υ_m, and υ_o between 0 and υ. In particular, this Doppler shift depends on υ_m even though there is no such dependence in case III—this contradicts the usual hand-waving derivation for the general shift as the product of all the multiplicative factors from each special case! This result for f' reduces to the usual textbook result for a moving source and observer when $\upsilon_m = 0$; furthermore, if you then take that textbook result and replace υ by $\upsilon + \upsilon_m$ you will recover the above. In addition, the boxed expression gives the well-known result that there is no Doppler shift when source and observer have equal velocity ($\upsilon_o = -\upsilon_s$) even if a wind is blowing.