A Semiclassical Derivation of Eigenenergies—C.E. Mungan, Summer 2008

Reference: C. Gianino, Phys. Educ. ***43****, 429 (2008).*

It is easy to derive the energy levels for three standard examples—formally solved using Schrödinger's equation—by invoking a few simple quantum mechanical and classical ideas.

The first case is a particle of mass m in an infinite one-dimensional box. If the box has length l, then the standing wave condition requires that there be nodes at the two walls. Therefore the wavelength λ_n of level n (where n = 1, 2, 3, …) is determined by

$$l = n\frac{\lambda_n}{2}\,. \tag{1}$$

Substituting the de Broglie relation in the form

$$\lambda_n = \frac{h}{m\upsilon_n}\,, \tag{2}$$

we find $\upsilon_n = nh/2ml$. Finally, since the particle's mechanical energy is purely kinetic, $E_n = m\upsilon_n^2/2$ and we thus obtain

$$\boxed{E_n = n^2 E_1 \quad \text{where} \quad E_1 = \frac{h^2}{8ml^2}}\,. \tag{3}$$

The second example is Bohr's model of the hydrogen atom. In level n (where n = 1, 2, 3, …) the electron orbits the proton in circles of radius r_n. The standing wave condition this time requires that an integral number of wavelengths fit around the circumference of an orbit,

$$2\pi r_n = n\lambda_n\,. \tag{4}$$

Substituting Eq. (2), we find

$$r_n = \frac{nh}{2\pi m\upsilon_n} \tag{5}$$

where the reduced mass m is very nearly equal to that of the electron. But the electrostatic force is centripetal, so that

$$\frac{e^2}{4\pi\epsilon_0 r_n^2} = m\frac{\upsilon_n^2}{r_n}\,. \tag{6}$$

Substitution of Eq. (5) leads to

$$\upsilon_n = \frac{e^2}{2\epsilon_0 hn}\,. \tag{7}$$

The total mechanical energy of the atom is

$$E_n = \frac{1}{2} m\upsilon_n^2 - \frac{e^2}{4\pi\epsilon_0 r_n} . \tag{8}$$

Using Eq. (6) to eliminate r_n gives $E_n = -m\upsilon_n^2 / 2$, which is negative since the orbits are bound. Finally substituting Eq. (7) results in

$$\boxed{E_n = \frac{E_1}{n^2} \quad \text{where} \quad E_1 = -\frac{me^4}{8\epsilon_0^2 h^2} = -13.6 \text{ eV}} . \tag{9}$$

The third example is a 1D simple harmonic oscillator of frequency *f* and mass *m*, whose motion is the one-dimensional projection of uniform circular motion where the radius equals the amplitude A_n in level *n*, so that Eq. (5) becomes

$$A_n = \frac{nh}{2\pi m\upsilon_n} = \frac{nh}{2\pi m(2\pi A_n f)} \quad \Rightarrow \quad A_n^2 = \frac{nh}{4\pi^2 mf} . \tag{10}$$

But the mechanical energy of the oscillator is $E_n = kA_n^2 / 2$ where *k* is the spring constant.[1] Substituting Eq. (10) and eliminating *k* using $2\pi f = \sqrt{k / m}$, we find $E_n = nhf / 2$. The final step is to note that *n* in Eq. (4) cannot take on all positive integral values this time. We have velocity nodes (or equivalently displacement antinodes) at the turning points, and velocity antinodes (or displacement nodes) at the equilibrium point. Therefore, a quarter cycle is like an organ pipe with one end open and the other end closed, so that we only get odd harmonics $n = 1, 3, 5, \ldots$. If we instead choose to label the oscillator levels by quantum numbers $n = 0, 1, 2, \ldots$ in the conventional manner, then their energies are

$$\boxed{E_n = (n + \tfrac{1}{2})hf} . \tag{11}$$

Unlike the particle in a box or the hydrogen atom, the energy levels are thus equally spaced with a gap between adjacent states of $\Delta E = hf$, which is Planck's relation for the emitted and absorbed photon energies, as required in the derivation of the blackbody equilibrium distribution.

---

[1]Equivalently, $E_n = m\upsilon_n^2 / 2$ since $k = m(2\pi f)^2$ and $\upsilon_n = 2\pi A_n f$.