One-Dimensional Elastic Collisions without Tears—C.E. Mungan, Fall 1997

An arbitrary 1D elastic collision in the absence of any net external force requires the simultaneous solution of the two equations

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2' \quad \text{and} \tag{1}$$

$$\tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 = \tfrac{1}{2} m_1 v_1'^2 + \tfrac{1}{2} m_2 v_2'^2 , \tag{2}$$

where we suppose that m_1, m_2, v_1, and v_2 are all known and that we are trying to solve for v_1' and v_2'. One approach is to solve Eq. (1) for v_1' and substitute it into Eq. (2) to get a quadratic equation for v_2'. Yuck!

There is a solution which avoids the quadratic equation. Let's rewrite Eqs. (1) and (2) as

$$m_1 (v_1 - v_1') = m_2 (v_2' - v_2) \quad \text{and} \tag{3}$$

$$m_1 (v_1^2 - v_1'^2) = m_2 (v_2'^2 - v_2^2) . \tag{4}$$

Now observe that both sides of Eq. (4) can be factored to give

$$m_1 (v_1 - v_1')(v_1 + v_1') = m_2 (v_2' - v_2)(v_2' + v_2) . \tag{5}$$

Divide Eq. (5) by Eq. (3) to obtain

$$v_1 + v_1' = v_2' + v_2 . \tag{6}$$

Equations (3) and (6) are now two simultaneous *linear* equations for v_1' and v_2', which are much easier to solve than quadratic equations. Namely, multiply Eq. (6) by m_1, add the result to Eq. (3), and rearrange to find the solution for v_2',

$$\boxed{v_2' = \frac{2 m_1 v_1 + (m_2 - m_1) v_2}{m_1 + m_2}} . \tag{7}$$

Then substitute this into Eq. (6) and rearrange to get the similar solution for v_1',

$$\boxed{v_1' = \frac{2 m_2 v_2 + (m_1 - m_2) v_1}{m_1 + m_2}} . \tag{8}$$

These are the final result and have the added advantage that we have already eliminated the solution of Eqs. (1) and (2) corresponding to the initial situation.