

An Electrostatics Problem—C.E. Mungan, Fall 1999

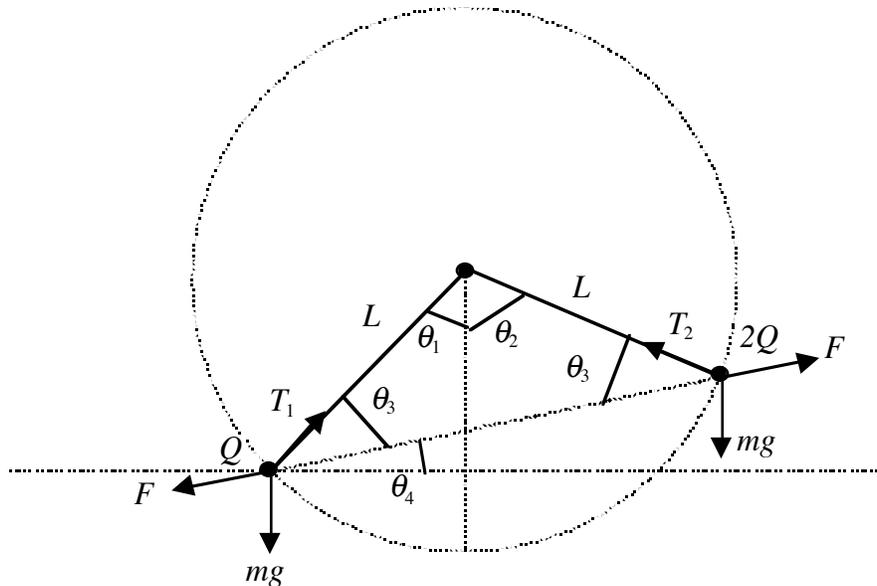
The following wonderful little problem was fully discussed and analyzed on the PHYS-L email list.

Serway's 4th edition of *Physics for Scientists and Engineers* problem 57 of chapter 23 states in part: Two small spheres of mass m are suspended from strings of length L that are connected at a common point. One sphere has charge Q ; the other has charge $2Q$. Assume that the angles θ_1 and θ_2 that the strings make with the vertical are small. How are θ_1 and θ_2 related?

Before reading further, try the problem yourself. It has a very simple solution. But then ask yourself this: Why the restriction to small angles?

Simple solution: Invoke the condition of rotational equilibrium to conclude that the center of mass of the system of two spheres must lie on a vertical line through the pivot point. Hence $\theta_1 = \theta_2$. It is now trivial to solve for the common tension, angle, electrostatic force, and anything else you like.

Another approach: The poser of the problem went on to ask whether it can be solved by invoking the condition of translational equilibrium alone? The answer is yes and it nicely shows why we require the restriction to small (in the sense of acute, it turns out) angles. Start with a general diagram,



where F is the electrostatic repulsive force between the two charges. The very hard way to do this problem is to set up a coordinate system with x and y horizontal and vertical, and to take components. Try it and see: Using the geometric fact that $\theta_4 = (\theta_2 - \theta_1)/2$ gives the messy equation

$$\cot \theta_1 - \cot \theta_2 = 2 \tan \left(\frac{\theta_2 - \theta_1}{2} \right) \quad (1)$$

after some elimination of unknowns among the four component equations. This is therefore a good example for students who always want to orient coordinate systems this way. A much better choice is to orient one axis of the coordinate system for each mass along the respective string. The perpendicular force balance then gives

$$mg \sin \theta_1 = F \sin \theta_3 \quad (2)$$

for the Q charge, and similarly

$$mg \sin \theta_2 = F \sin \theta_3 \quad (3)$$

for the $2Q$ charge. Equating the left-hand sides of Eqs. (2) and (3) implies

$$\sin \theta_1 = \sin \theta_2. \quad (4)$$

This has two solutions:

(i) $\theta_1 = \theta_2$ as before; or

(ii) $\theta_1 = 180^\circ - \theta_2$, which implies that one sphere is directly across the circle from the other, thus putting the center of mass at the pivot, as is acceptable albeit unstable to rotations of the system. Of course this can only happen if F has a very large value, so that the string does not collapse. This solution is however ruled out by the small-angle restriction, which thus does not necessarily imply the small-angle approximation that you may have thought of from the problem wording.

Comments: Notice that the above two solutions are consistent with Eq. (1) and further that the small-angle approximation to Eq. (1) does give solution (i). Nevertheless I am unable to prove that (i) and (ii) are the general solutions to Eq. (1). Can you? I suspect that a graphical proof is needed, as trig identities only gave me a huge mess.

It is also worth noting in hindsight that both solutions are in fact given by our initial simple solution. After all, the symmetry of both the gravitational force and the Coulombic force merely require that the horizontal distance from one mass to a vertical line through the pivot be equal and opposite to that of the other mass. However, the more formal alternative approach spares one from drawing hasty conclusions from symmetry, as one might easily be led to do. Shame, shame!