Area of an Ellipse in Polar Coordinates—C.E. Mungan, Fall 2017

Consider an ellipse centered on the origin and with the *x* and *y* axes aligned along the semi-major axis *a* and the semi-minor axis *b*, respectively, so that the equation of the ellipse in rectangular coordinates is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1. \tag{1}$$

Convert to polar coordinates by substituting into it

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta \tag{2}$$

to obtain

$$r(\theta) = \frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}. \tag{3}$$

This result can be inserted into the formula for the area of the ellipse to get

$$A = \int_0^{2\pi} d\theta \int_0^{r(\theta)} r\,dr = \frac{a^2b^2}{2}\int_0^{2\pi} \frac{d\theta}{b^2\cos^2\theta + a^2\sin^2\theta}. \tag{4}$$

The remaining integral can be done by making the change of variables from θ to ϕ given by

$$a\tan\theta = b\tan\phi \quad \Rightarrow \quad a\sec^2\theta d\theta = b\sec^2\phi d\phi, \tag{5}$$

as can be made more obvious by rewriting Eq. (4) as

$$A = \frac{a^2b^2}{2}\int_0^{2\pi} \frac{\sec^2\theta d\theta}{b^2 + a^2\tan^2\theta} \tag{6}$$

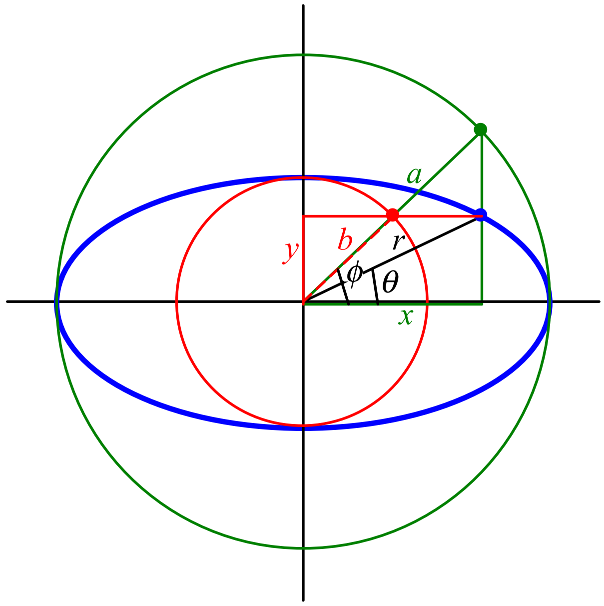
and using the Pythagorean identity $1 + \tan^2\phi = \sec^2\phi$. The result immediately simplifies to the standard answer

$$A = \frac{a^2b^2}{2}\int_0^{2\pi} \frac{(b/a)d\phi}{b^2} = \pi ab. \tag{7}$$

The angle ϕ introduced in Eq. (5) is known as the *eccentric anomaly* and implies that

$$x = a\cos\phi \quad \text{and} \quad y = b\sin\phi \tag{8}$$

to be carefully distinguished from Eq. (2). Equation (8) follows from two facts: (i) the ratio of *y* to *x* in Eq. (8) when equated to the ratio of *y* to *x* in Eq. (2) agrees with Eq. (5); and (ii) Eq. (8) satisfies Eq. (1). This result can be geometrically interpreted in terms of two right triangles and two circles, one pair drawn in green and the other pair drawn in red on the diagram at the top of the next page.



Consider an arbitrary point in blue on the ellipse at rectangular coordinates (x, y) or polar coordinates (r, θ). The green circle is centered on the origin with radius *a*. Draw a perpendicular to the *x* axis upward through the blue point on the ellipse until it meets the green circle at the green point. Then connect the origin to that green point by the indicated green radial line. Likewise the red circle is centered on the origin with radius *b*. Draw a perpendicular to the *y* axis rightward so that it intersects the red circle at the red point and ends at the blue point on the ellipse. Again connect the origin to that red point by the indicated red radial line (which by construction must overlap the green radial line). Inspection of the side lengths of the resulting green and red right triangles shows that they agree with Eq. (8) and thereby illustrates the physical meaning of angle ϕ. Imagine the blue point as being a planet orbiting counter-clockwise around the origin. When it is on the *x* axis, so are the red and green points, i.e., $\theta = \phi = 0$. As the planet proceeds into quadrant I, it lags behind the green and red points so that $\theta < \phi$, but catches up to them along the *y* axis where $\theta = \phi = \pi / 2$. It then surpasses them in quadrant II so that $\theta > \phi$, but all the points again meet along the –*x* axis where $\theta = \phi = \pi$. The behavior in quadrants III and IV then repeats that of I and II, respectively, except with π added to all angles.