Mechanical Equilibrium of a Rigid Body—C.E. Mungan, Spring 2023

If exactly one nonzero external force acts on an extended rigid body, it cannot be in equilibrium. If exactly two forces act on it, we get equilibrium if and only if the two forces are equal & opposite and are directed along the same line of action. If four or more forces act on the body, we can choose an origin and then combine all forces $\mathbf{F}_i$ with $i \ge 3$ into a single new force that equals the sum of these forces and whose torque equals the sum of their torques [1]. So we are left with the question of when will exactly three forces[1] result in equilibrium? The answer [2] is that the forces must satisfy two conditions: (1) they must vectorially sum to zero, and (2) the three forces must be *concurrent* which means that their three lines of action must intersect at a common point (possibly at infinity).

Equilibrium here means that two facts are true about the body. First, its center of mass will have zero translational acceleration. That will be true if and only if

$$\sum_{i=1}^{3} \mathbf{F}_i = 0\,. \tag{1}$$

Second, its angular acceleration about its center of mass (defining an origin O) must be zero, which requires

$$\sum_{i=1}^{3} \mathbf{r}_i \times \mathbf{F}_i = 0 \tag{2}$$

where $\mathbf{r}_i$ is the position vector from O to the point of application of force $\mathbf{F}_i$. We can prove that if both Eqs. (1) and (2) hold, then it also must be true that

$$\sum_{i=1}^{3} \mathbf{r}'_i \times \mathbf{F}_i = 0 \tag{3}$$

where $\mathbf{r}'_i$ is the position vector from any other origin O' to the point of application of force $\mathbf{F}_i$. In other words, we can shift the origin by $\mathbf{R}$ for the calculation of the torques from O to O' such that

$$\mathbf{r}_i = \mathbf{r}'_i + \mathbf{R} \tag{4}$$

for all $i = 1$ to 3. The proof is straightforward. Substitute Eq. (4) into (2) to get

$$\sum_{i=1}^{3} \mathbf{r}'_i \times \mathbf{F}_i + \mathbf{R} \times \sum_{i=1}^{3} \mathbf{F}_i = 0\,. \tag{5}$$

But the second sum is zero according to Eq. (1), and thus Eq. (3) follows. There is nothing special about having chosen the origin O to coincide with the center of mass of the body; we could have started with any origin O we like. The key result is that if Eqs. (1) and (2) hold for

[1]If the object rotates about an axle, the axle is assumed to exert only a single reaction force on the object. For example, there cannot be a frictional force distributed around the circumference of the axle, nor a normal force distributed along the length of the axle.

any *one* choice of origin, then they must hold for *all* possible choices of origin. We will apply this conclusion to concurrency.

First let's show that equilibrium requires the three forces to be coplanar. If any two force vectors were not contained in the same plane, then we could choose the origin to be located at the point of application of the third force so that it produces zero torque. But the two noncoplanar forces would produce noncoplanar torques about that origin, and thus the sum of the three torques would not be zero. Therefore, any two of the forces must be coplanar. In addition, the third force must equal the negative of the vector sum of those two forces, and thus the third force must also be in the same plane as them, so that all three forces are necessarily coplanar.

Now start the concurrency proof by supposing that at least two of the three equilibrium forces are not parallel to each other. In that case, they must intersect at one specific point C in their common plane. Choose that point to be the origin so that neither of those two forces produces a torque about it. Then the only way the three torques can sum to zero is if the third force also has a line of action passing through that point of application C, which is therefore concurrent to all three forces.

Finally, a special case is that C could be located at infinity, which happens if all three forces are parallel to each other [2]. Choose that parallel direction to define the *y* axis. Now note that, for any choice of origin, the torque produced by a force **F** is unchanged if we translate the force along its line of action. (Originally the torque is $\mathbf{r}\times\mathbf{F}$. Replacing **r** by $\mathbf{r}+m\mathbf{F}$, where *m* is any numerical multiple we like with units of m/N, gives exactly the same torque since $\mathbf{F}\times\mathbf{F}=0$.) So, translate all three parallel forces such that their points of application all lie along the *x* axis in their common *xy* plane. Equilibrium now requires that

$$\sum_{i=1}^{3} x_i F_{yi} = 0\,. \qquad (6)$$

This sum reduces to at most two terms if the origin is chosen to pass through the line of action of one of the forces. An application is to a one-dimensional rigid rod like a see-saw.

[1] K.L. Goh, "Concurrent and coplanar forces that are in equilibrium," *Phys. Teach.* **56**, 384 (2018).

[2] A. Alameh, "Static equilibrium in a uniform gravitational field," *Phys. Teach.* **61**, 298 (2023).