Extrema & Saddle Points for a Function of Two Variables—C.E. Mungan, Fall 2008

Suppose we wish to find the extrema and saddle points of $f(x,y)$. Begin by simultaneously solving the two equations

$$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0 \tag{1}$$

to find all of the points (x_s, y_s) that could be an extremum or saddle point. Next, for each of these suspect points in turn, evaluate all of the second partial derivatives,

$$f_{xx} \equiv \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x_\text{s}, y_\text{s})} \quad \text{and} \quad f_{yy} \equiv \left.\frac{\partial^2 f}{\partial y^2}\right|_{(x_\text{s}, y_\text{s})} \quad \text{and} \quad f_{xy} \equiv \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(x_\text{s}, y_\text{s})} = f_{yx}\,. \tag{2}$$

Now compute the determinant of these numerical values for the second partials,

$$D \equiv \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}. \tag{3}$$

If $D < 0$ then (x_s, y_s) is a saddle point. If $D = 0$ then our test is indeterminate; either go on to evaluate higher derivatives or preferably graph the function and look at it to see what is happening at point (x_s, y_s). Finally, if $D > 0$ then (x_s, y_s) is an extremum; it is a maximum if f_{xx} and f_yy are both negative, and it is a minimum if both unmixed second partials are positive.

Footnotes:

1. Equation (3) should NOT contain variables x or y in it. You need to substitute the numerical values of x_s and y_s into the second partial derivatives in Eq. (2), so that f_{xx} and the rest of them become pure numbers.

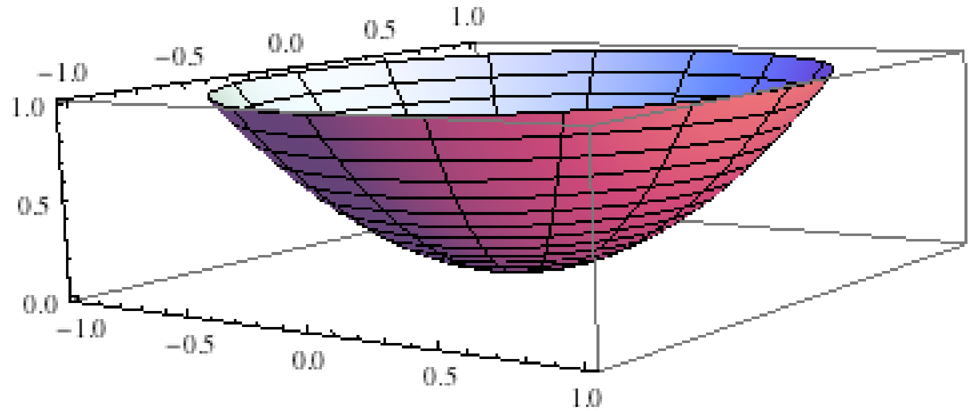
2. It is impossible for D to be positive if either of the following conditions hold:

(i) f_{xx} or f_{yy} or both of them are zero;

(ii) neither f_{xx} nor f_{yy} is zero, but they have opposite signs.

You should prove this claim! It can be easily done by writing out the determinant in Eq. (3) and noting that $f_{yx} = f_{xy}$.

3. Do not try to memorize which signs of D and of the second partials correspond to which final result, because it's too easy to get them wrong. Instead consider some simple example where you know the answers, such as the paraboloid $f(x,y) = x^2 + y^2$ plotted on the next page. Obviously it has a minimum at the origin. Equation (2) gives $f_{xx} = f_{yy} = 2$ and $f_{xy} = f_{yx} = 0$. Then Eq. (3) gives $D = 4$. So clearly the right signs must be: $D > 0$ for an extremum, and f_{xx} and f_yy both positive for a minimum.



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