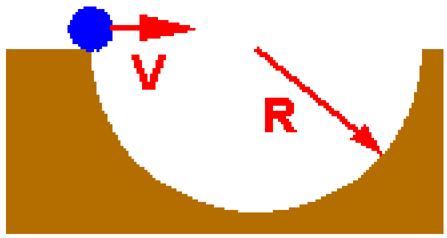
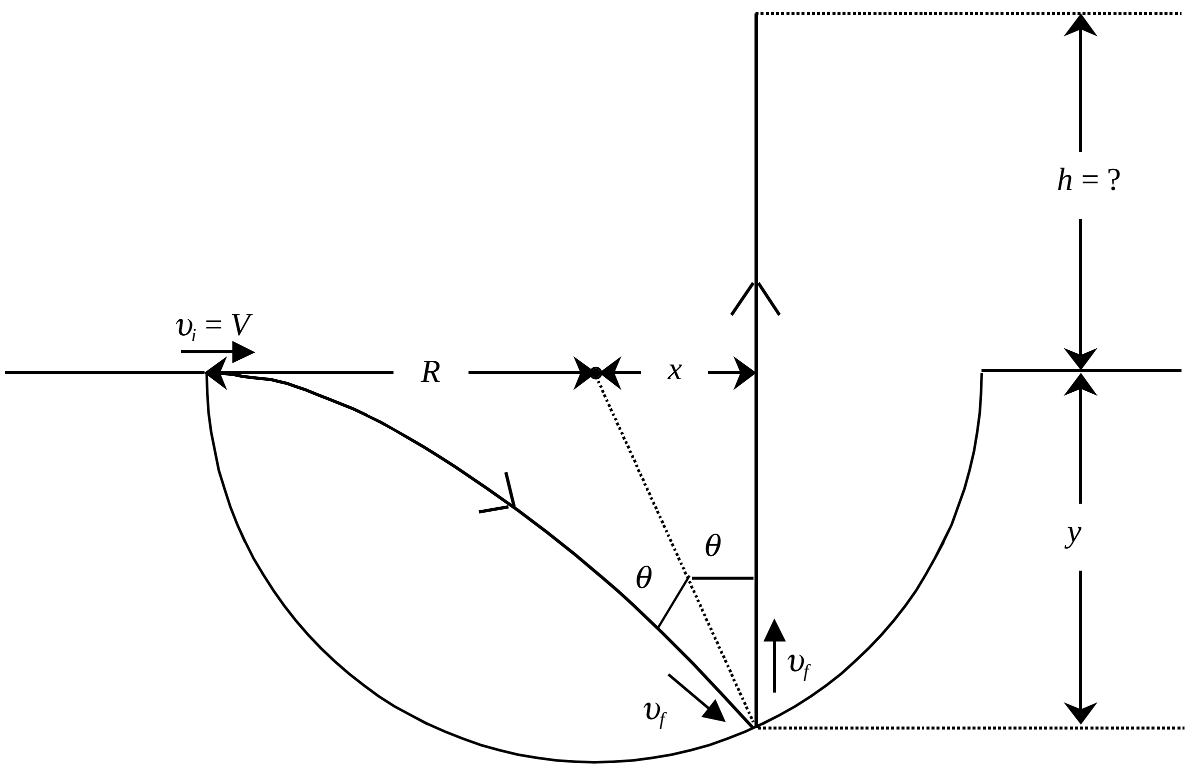
Falling Ball Puzzle—C.E. Mungan, Spring 2000

The following puzzle is due to Dr. Akaske, formerly at Hamline University. The little ball of mass *m* is launched horizontally with initial velocity *V* from the lip of the semi-cylindrical depression of radius *R*. The ball makes a perfectly elastic collision with the depression and is observed to rise straight up. How high will it rise above the lip of the depression? Your answer is to be expressed in terms of *R* only. (Consider the ball to be a point mass.)



Solution:

It is logical to choose a coordinate system whose origin is at the center of the cylinder with positive *x* running horizontally to the right and positive *y* running vertically downward,



where the fact that the bounce is elastic implies that the two angles marked θ and that the two speeds marked υ_f are equal in each case. From conservation of energy we see that the height of interest is

$$h = \frac{V^2}{2g}. \tag{1}$$

By taking the components of the final velocity just before hitting the cylinder, we get

$$\tan(2\theta) = \frac{\upsilon_{fx}}{\upsilon_{fy}} = \frac{V}{\sqrt{2gy}} = \frac{2\tan\theta}{1-\tan^2\theta} \tag{2}$$

where I used the standard equations of kinematics to get the second equality, and a well-known trig identity to get the last equality. On the other hand, we also see from the diagram that

$$\tan\theta = \frac{x}{y} = \frac{\sqrt{R^2 - y^2}}{y} = \sqrt{(R/y)^2 - 1}, \tag{3}$$

making use of the equation of a circle to deduce the second equality. Substituting Eq. (3) into the fourth term and Eq. (1) into the third term of Eq. (2) results in

$$\sqrt{\frac{h}{y}} = \frac{2\sqrt{(R/y)^2 - 1}}{2 - (R/y)^2}. \tag{4}$$

Next, we consider the parabolic trajectory to get a second expression for h. Eliminate the time of flight t between $R + x = Vt$ and $y = gt^2/2$ to get $y = (R+x)^2/4h$ using Eq. (1). Again utilizing the equation of a circle, this can be rearranged as

$$h = \frac{\left(R + \sqrt{R^2 - y^2}\right)^2}{4y}. \tag{5}$$

Substitute this expression into the left-hand side of Eq. (4). The result is conveniently expressed in terms of the dimensionless variables $X \equiv x/R$ and $Y \equiv y/R$, where $0 < X < 1$ and $0 < Y < 1$ by inspection of the diagram, as

$$1 + \sqrt{1 - Y^2} = \frac{4\sqrt{1-Y^2}}{2 - 1/Y^2}. \tag{6}$$

Since $X = \sqrt{1 - Y^2}$ from the equation of a circle, Eq. (6) can be simplified to

$$2X^2 - 4X + 1 = 0. \tag{7}$$

Putting this into the quadratic formula, one finds a single solution with $0 < X < 1$, namely

$$\frac{x}{R} = 1 - \frac{1}{\sqrt{2}} = 0.293 \tag{8a}$$

and thus

$$\frac{y}{R} = \sqrt{1 - \left(\frac{x}{R}\right)^2} = \left(\sqrt{2} - 0.5\right)^{1/2} = 0.956. \tag{8b}$$

This can be substituted into Eq. (5) which can be rewritten as

$$\frac{h}{R} = \frac{(1+X)^2}{4Y} = \frac{\left(\sqrt{2}-0.5\right)^{3/2}}{2} = \tfrac{1}{2}Y^3 = 0.437, \tag{9}$$

i.e. the ball rises to 44% of the radius of the cylinder. This completes the solution. The angle with which the ball strikes the cylinder relative to the radial direction is $\theta = \tan^{-1}(X/Y) = 17.0^\circ$, or equivalently 55.9° relative to the horizontal.

What is remarkable about this solution is that it is independent of *m*, *g*, and *V* (provided all are positive). The size *R* of the cylinder already incorporates the latter two factors. To look at it from another point of view, for a given cylinder of fixed size and assuming standard gravity of $g = 9.80$ m/s^2, the initial velocity of the ball must be chosen to be $V = 2.93\sqrt{R}$ with *R* in m and *V* in m/s.