Four Infinite Sums of Reciprocal Squares and Quartics—C.E. Mungan, Spring 2021

In this paper I want to derive the four sums

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^4 + a^4} = \frac{\pi}{\sqrt{2}a^3} \frac{\sinh\sqrt{2}\pi a + \sin\sqrt{2}\pi a}{\cosh\sqrt{2}\pi a - \cos\sqrt{2}\pi a}, \tag{1}$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^4 - a^4} = \frac{-\pi}{2a^3}(\cot\pi a + \coth\pi a), \tag{2}$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a}\coth\pi a, \text{ and} \tag{3}$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 - a^2} = \frac{-\pi}{a}\cot\pi a. \tag{4}$$

These can all be checked in WolframAlpha. For instance, enter "sum 1/(n^4–a^4) from –inf to inf" to get Eq. (2). Knowing any one of these sums, we can get the other three. For example, replacing *a* with *ia* will interconvert between Eqs. (3) and (4). Likewise replacing *a* with $a(1+i)/2^{1/2}$ will do so for Eqs. (1) and (2). Finally to interconvert between the first and second pair of equations, use the partial fraction decomposition

$$\frac{1}{n^4 - a^4} = \frac{1}{2a^2}\left[\frac{1}{n^2 - a^2} - \frac{1}{n^2 + a^2}\right]. \tag{5}$$

Thus I only need to derive one of the sums. With the help of Math Stack Exchange, I will derive Eq. (4) using some neat ideas. I will choose not to use residues or summation of Fourier coefficients, which are two other powerful ways to derive it, but are a little less accessible to undergraduates.

But first let's check Eq. (4) against the standard result for the special case of $a = 0$ using l'Hôpital's rule. To avoid a divergence at $n = 0$, let's split off that term and rewrite Eq. (4) as

$$\frac{-1}{a^2} + 2\sum_{n=1}^{\infty} \frac{1}{n^2 - a^2} = \frac{-\pi}{a}\cot\pi a \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{n^2 - a^2} = \frac{1 - \pi a\cot\pi a}{2a^2}. \tag{6}$$

Now Taylor expand cotangent written as cosine divided by sine, each to third order, to correctly get

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \lim_{a\to 0} \frac{1 - \pi a \dfrac{1 - (\pi a)^2/2}{\pi a - (\pi a)^3/6}}{2a^2} = \lim_{a\to 0} \frac{1 - [1 - (\pi a)^2/2][1 + (\pi a)^2/6]}{2a^2} = \frac{\pi^2}{6}. \tag{7}$$

Proceeding to a derivation of Eq. (4), let's use Euler's expansion of the sinc function as an infinite product of its roots. A formal proof of this decomposition is a bit involved, but the idea can be easily motivated by way of example. The cubic polynomial $f(x) = 2x^3 - 9x^2 + 7x + 6$ has the three roots 3, 2, and –1/2. Furthermore it is equal to 6 when $x = 0$. Thus we can write the polynomial as

$$f(x) = 6\left(1-\frac{x}{3}\right)\left(1-\frac{x}{2}\right)\left(1+\frac{x}{1/2}\right). \tag{8}$$

Assuming we can do the same for functions (thinking of their Taylor expansions as a polynomial of infinite order) and not just for finite-order polynomials, consider $\operatorname{sinc}(\pi a) \equiv \sin(\pi a)/(\pi a)$ as a function of *a*, which has roots at $\pm 1, \pm 2, \ldots$ and which is equal to 1 at $a = 0$. We thus immediately obtain

$$\frac{\sin \pi a}{\pi a} = \prod_{n=1}^{\infty}\left(1-\frac{a^2}{n^2}\right). \tag{9}$$

An immediate application of this formula is the Wallis infinite product for π obtained by substituting $a = 1/2$ and taking the reciprocals of both sides.

Next take the natural logarithm of both sides of Eq. (9) to get

$$\ln \sin \pi a - \ln \pi a = \sum_{n=1}^{\infty} \ln\left(1-\frac{a^2}{n^2}\right). \tag{10}$$

Now differentiate both sides with respect to *a* to find

$$\pi \frac{\cos \pi a}{\sin \pi a} - \frac{1}{a} = -\sum_{n=1}^{\infty} \frac{2a/n^2}{1-a^2/n^2} \tag{11}$$

and finally divide through by $-2a$ to obtain

$$\frac{-\pi}{2a}\cot \pi a + \frac{1}{2a^2} = \sum_{n=1}^{\infty} \frac{1}{n^2 - a^2} \tag{12}$$

which is Eq. (6) and thereby completes the proof.