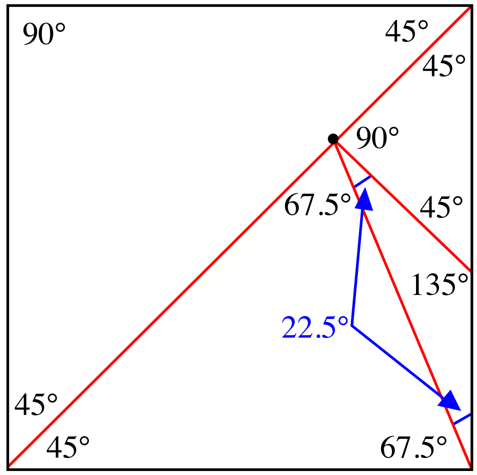
Tile a Square With Four Nonidentical Isosceles Triangles—C.E. Mungan, Summer 2018

Divide a square into four isosceles triangles, no two of which are congruent to each other:



If each side of the square has unit length, you can easily find the length of each side of these four triangles. To find this solution, I first established this theorem: Given a triangle which has an internal angle whose measure is triple that of another internal angle, you can always divide it into two isosceles triangles. [Proof: Call the two angles θ and 3θ. Trisect the latter into angles θ and 2θ, with the smaller of those two angles on the same side of the trisector as the original internal angle θ. You then end up with two isosceles triangles, one with two interior angles of θ and the other with two interior angles of 2θ.] So take a square as in the following diagram, draw diagonal CD, then draw horizontal line AE starting from an arbitrary point A along that diagonal. Now slide point A along the diagonal until angles CAB and ABC become equal. Since angle ACB is 45°, that happens when CAB and ABC are 67.5°. But now notice that triangle ABE has interior angles BAE of 67.5° and ABE of 22.5°. Thus from our theorem, we can trisect angle BAE by creating line AF such that angle FAB is 22.5°. We have now tiled the square with five isosceles triangles, such that two of them (AEF and ADE) are congruent 45-45-90 triangles. Erasing line AE now solves the puzzle.