Freefall due to Refraction of Matter Waves by Gravity—C.E. Mungan, Summer 2021

The authors of AJP **89**, 634 (2021) present three demonstrations that the de Broglie wave of a particle in a gravitational field refracts toward regions of lower gravitational potential because time flows slower in such regions. That causes the freefall of the particle. I summarize the third demonstration, which is a good approach for a first course in modern physics.

The demonstration is in three steps. First, one needs the formula for gravitational time dilation. Consider two clocks that are at rest above the same point on earth's surface, one at higher altitude by a height H that is small compared to earth's radius, so that the magnitude g of the gravitational field can be considered equal for the two clocks. A photon that falls from the higher to the lower clock will be gravitationally blue-shifted in its frequency f by[1]

$$\Delta f = f \frac{\Delta V}{c^2} \tag{1}$$

where c is the speed of light and $\Delta V = gH$ is the change in gravitational potential between the two locations. (See the Appendix for a quick derivation of this result.) Using the frequency of light to count cycles of time t, Eq. (1) can be rewritten as

$$\Delta t = t \frac{gH}{c^2} \tag{2}$$

because the period is $T = 1/f \Rightarrow |df/f| = |dT/T|$ and $t = NT \Rightarrow dt/t = dT/T$ is time measured for N (not necessarily integral) periods. In other words, if the time interval measured by the lower clock is t, then the corresponding time interval measured by the upper clock will be greater by Δt. Time runs faster in weaker gravity.

Second, we need the formula for the phase speed of a matter wave. Phase speed is given by[2]

$$\upsilon_\text{p} = \frac{\omega}{k} = \frac{\hbar\omega}{\hbar k} = \frac{E}{p} = \frac{\gamma mc^2}{\gamma m\upsilon_x} = \frac{c^2}{\upsilon_x} \tag{3}$$

for a particle of mass m traveling horizontally at speed υ_x. Consider one harmonic plane wave component of a matter wavepacket. Its phase front is perpendicular to the direction of its velocity. So if at one point in time the particle is traveling horizontally (as indicated by the black arrow in Fig. 1) then the wavefront is vertical (as indicated by the black line segment). But if time flows faster for the top end of the wavefront, it will advance horizontally relative to the lower end so that it will tilt (as indicated by the red line segment) which implies the particle's velocity vector will now point diagonally downward (as indicated by the red arrow).

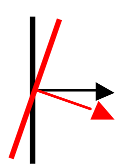

Figure 1

This effect is exactly like the mirage effect which causes a light wave descending downward from the blue sky toward a hot highway (where the hot air has lower refractive index and hence larger speed of light) to deflect upward.[3]

Third, the AJP paper use these two results to consider the motion in the *xy*-plane of a particle launched horizontally from height H above earth's surface at initial speed υ_x as sketched in Fig. 2.

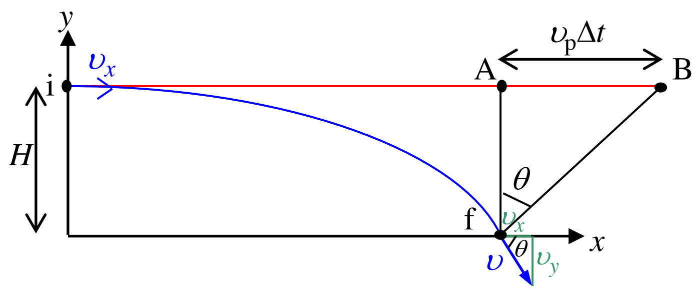

Figure 2

The particle starts at initial point i. In the absence of gravity, it would move to point A in a time t. However, with a vertically downward uniform gravitational field, the top end of the matter wavefront will instead advance to point B. From the geometry of the triangle connecting the particle's final point f to points A and B, we see that

$$\tan\theta = \frac{\upsilon_p \Delta t}{H} = \frac{gt}{\upsilon_x} \tag{4}$$

after substituting Eqs. (2) and (3). On the other hand, the lower velocity triangle indicates that

$$\tan\theta = \frac{\upsilon_y}{\upsilon_x}. \tag{5}$$

Equating Eqs. (4) and (5) leads to the key result

$$\upsilon_y = gt \tag{6}$$

indicating that freefall kinematics follows from gravitational refraction of matter waves.

[1]C.E. Mungan, "Relativistic effects on clocks aboard GPS satellites," Phys. Teach. **44**, 424 (Oct. 2006), Eq. (2).
[2]https://www.usna.edu/Users/physics/mungan/_files/documents/Scholarship/MatterWaves.pdf
[3]https://www.usna.edu/Users/physics/mungan/_files/documents/Scholarship/HighwayMirages.pdf

Appendix: Brief Derivation of Eq. (1)

Suppose a photon of frequency f falls vertically downward through a height H in a uniform gravitational field g. It must gain (kinetic) energy $E = hf$ equal to the lost gravitational potential energy $U = mgy$,

$$\Delta(hf) = \Delta(mgy) = m\Delta(gy) = \frac{E}{c^2}\Delta V = \frac{hf}{c^2}\Delta V. \tag{7}$$

Canceling Planck's constant h on both sides immediately gives Eq. (1). This derivation is a bit of a cheat in that we are saying the energy of a photon is purely kinetic because it has no rest mass *while at the same time* claiming it has potential energy associated with its relativistic mass m.