Fresnel Boundary Conditions—C.E. Mungan, Fall 1998

Problem: Given that

$$A_1 e^{ik_1 x} + A_2 e^{ik_2 x} = A_3 e^{ik_3 x} \tag{1}$$

for all x, prove that $A_1 + A_2 = A_3$ and that $k_1 = k_2 = k_3$.

Solution: Substitute $x = 0$ into Eq. (1) to find

$$\boxed{A_1 + A_2 = A_3}. \tag{2}$$

Next differentiate Eq. (1) and then substitute $x = 0$ into the result to get

$$k_1 A_1 + k_2 A_2 = k_3 A_3. \tag{3}$$

Repeat this process once more to find

$$k_1^2 A_1 + k_2^2 A_2 = k_3^2 A_3. \tag{4}$$

Now substitute (2) into (3) and rearrange to obtain

$$A_1(k_1 - k_3) = A_3(k_3 - k_2) \tag{5}$$

and also put (2) into (4) to give

$$A_1(k_1^2 - k_3^2) = A_2(k_3^2 - k_2^2). \tag{6}$$

Now divide Eq. (6) by (5) to find

$$k_1 + k_3 = k_3 + k_2, \text{ i.e., } \boxed{k_1 = k_2} \tag{7}$$

and then substitute both Eqs. (7) and (2) into (3) to deduce that

$$k_2 = k_3 \text{ and therefore } \boxed{k_1 = k_3}. \tag{8}$$

Application: Let A_1 be either E_{0i}^{tan} or H_{0i}^{tan}, A_2 be either E_{0r}^{tan} or H_{0r}^{tan}, and A_3 be either E_{0t}^{tan} or H_{0t}^{tan}, i.e., the tangential components of the incident, reflected, or transmitted electric or magnetic field amplitudes. Also let k_1 be $k_i \sin\theta_i$, k_2 be $k_r \sin\theta_r$, and k_3 be $k_t \sin\theta_t$, i.e., the x-components of the relevant propagation vectors, where the coordinate system is chosen so that the origin is on the interface with the x-axis parallel to the interface in the plane of incidence. Note that these definitions imply for example that $\mathbf{k}_i \cdot \mathbf{r} = k_1 x$ for any point on the interface. Equation (1) is now a statement of the boundary conditions that the tangential component of the total electric or magnetic field is continuous across an interface.

The magnitudes of the propagation vectors are just the propagation numbers,

$$k_i = n_i \frac{2\pi}{\lambda_0} = k_r \text{ and } k_t = n_t \frac{2\pi}{\lambda_0} \tag{9}$$

where n_i and n_t are the indices of refraction in the incident and transmitted media, and λ_0 is the vacuum wavelength of the light wave. Thus Eq. (7) implies

$$\boxed{\theta_i = \theta_r} \tag{10}$$

which is the Law of Reflection, while Eq. (8) leads to

$$\boxed{n_i \sin\theta_i = n_t \sin\theta_t} \tag{11}$$

which is Snell's Law of Refraction. Finally, Eq. (2) becomes

$$\boxed{E_{0i}^{tan} + E_{0r}^{tan} = E_{0t}^{tan} \text{ and } H_{0i}^{tan} + H_{0r}^{tan} = H_{0t}^{tan}} \tag{12}$$

so that the tangential components of the amplitudes, and not just of the total fields, are continuous across the interface. Equation (12) is used to derive the Fresnel equations.