Thermally Induced Agitation of a Simple Pendulum—C.E. Mungan, Spring 2000

The generalized equipartition theorem is

$$\overline{E} = \frac{1}{n} kT\left(2 - \delta_{a0} - \delta_{b0}\right) \tag{1a}$$

provided the energy per conjugate pair of degrees of freedom q and p can be written in the form

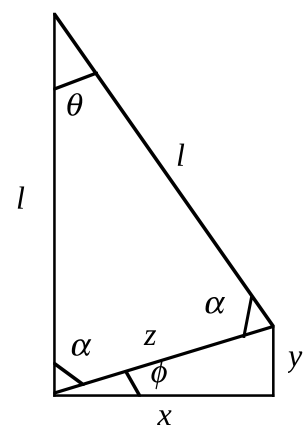
$$E \equiv KE + PE = a|p|^n + b|q|^n . \tag{1b}$$

where $n > 0$. The absolute value bars in Eq. (1b) ensure that the energy is always positive. The Krönecker delta functions in Eq. (1a) reduce the average energy if some degree of freedom is either frozen out or otherwise not present. For example, three common applications of Eq. (1) are:

- A classical free particle for which $KE = p^2/2m$ and $PE = 0$. In this case $n = 2$ and $b = 0$, so that $\overline{E} = kT/2$.
- A photon for which $KE = pc$ and $PE = 0$. In this case $n = 1$ and $b = 0$, so that $\overline{E} = kT$.
- A simple harmonic oscillator for which $KE = p^2/2m$ and $PE = k_s q^2/2$. In this case $n = 2$, so that $\overline{E} = kT$.

A derivation of Eq. (1) is in the Appendix.

To apply this to a simple pendulum, whose geometry is sketched below, we can consider using any of three different choices for our conjugate pair of variables.



We see that

$$2\alpha + \theta = 180^\circ \quad \Rightarrow \quad \alpha = 90^\circ - \theta/2 \quad \Rightarrow \quad \phi = 90^\circ - \alpha = \theta/2 \tag{2a}$$

and

$$z = 2l\sin(\theta/2) \tag{2b}$$

so that

$$x = z\cos\phi \cong l\theta \quad \text{and} \quad y = z\sin\phi \cong \tfrac{1}{2} l\theta^2 \quad \Rightarrow \quad y \cong x^2/2l \tag{2c}$$

assuming small angle θ.

One choice of conjugate variables is θ and the angular momentum $L \equiv I\dot{\theta}$,

$$KE = \tfrac{1}{2} I\dot{\theta}^2 = L^2 / 2I \quad \text{and} \quad PE = mgy \cong \tfrac{1}{2} mgl\theta^2, \tag{3}$$

while another is x and $p_x \equiv m\dot{x}$,

$$KE = \tfrac{1}{2} I\dot{\theta}^2 \cong p_x^2 / 2m \quad \text{and} \quad PE = mgy \cong mgx^2 / 2l. \tag{4}$$

In obtaining the kinetic energy in Eq. (4), I differentiated Eq. (2c) for x and used $I \equiv ml^2$. However, since

$$KE = p^2 / 2m = (p_x^2 + p_y^2) / 2m,$$

this is equivalent to putting $p_y \equiv m\dot{y} \cong 0$. Hence, choosing y and p_y as the conjugate variables implies

$$KE = p_y^2 / 2m \cong 0 \quad \text{and} \quad PE = mgy. \tag{5}$$

Now applying Eq. (1) to any of Eqs. (3)–(5) in all cases gives the same answer, namely

$$\overline{E} = kT \tag{6}$$

in agreement with the third application discussed above.

Appendix—Derivation of the Generalized Equipartition Theorem

The average energy is given by

$$\overline{E} = \frac{\sum_{\text{states } s} E_s e^{-\beta E_s}}{\sum_{\text{states } s} e^{-\beta E_s}}. \tag{7}$$

We can convert the summations into integrals by introducing the density of states g,

$$\sum_{\text{states } s} \rightarrow \int_{-\infty}^{+\infty} g(q,p)\, dq dp. \tag{8}$$

But according to Heisenberg's Uncertainty Principle, each quantum state occupies a volume of h in q–p phase space, so that $g = 1/h$. Using Eq. (1b), Eq. (7) thus becomes

$$\overline{E} = \overline{KE + PE} = \frac{\int_0^\infty \int_0^\infty ap^n e^{-\beta ap^n} e^{-\beta bq^n}\, dpdq + \int_0^\infty \int_0^\infty bq^n e^{-\beta ap^n} e^{-\beta bq^n}\, dpdq}{\int_0^\infty \int_0^\infty e^{-\beta ap^n} e^{-\beta bq^n}\, dpdq} \tag{9}$$

where I canceled a factor of $4/h$ in front of each integral. This can be further simplified to

$$\overline{E} = \frac{\int_0^\infty a p^n e^{-\beta a p^n}\, dp}{\int_0^\infty e^{-\beta a p^n}\, dp} + \frac{\int_0^\infty b q^n e^{-\beta b q^n}\, dq}{\int_0^\infty e^{-\beta b q^n}\, dq} = \beta^{-1}\frac{I_1}{I_2}(2-\delta_{a0}-\delta_{b0}) \tag{10}$$

where I put

$$I_1 \equiv \int_0^\infty x^n e^{-x^n}\, dx \quad \text{and} \quad I_2 \equiv \int_0^\infty e^{-x^n}\, dx \tag{11}$$

by defining $x \equiv (\beta a)^{1/n} p$ in the first ratio of integrals and $x \equiv (\beta b)^{1/n} q$ in the second, with the delta functions allowing for the possibilities that either *a* or *b* is zero. We now simply integrate I_1 by parts,

$$\begin{aligned} &u = x \Rightarrow du = dx \quad \text{and} \quad dv = x^{n-1}e^{-x^n}\, dx \Rightarrow v = -\frac{1}{n}e^{-x^n} \\ &\therefore I_1 = -\frac{x}{n}e^{-x^n}\Big|_0^\infty + \frac{1}{n}I_2 = \frac{1}{n}I_2 \end{aligned} \tag{12}$$

where I used l’Hôpital’s rule to evaluate the term at ∞ with $n > 0$:

$$y \equiv \frac{x}{e^{x^n}} \quad \Rightarrow \quad \ln y = \ln x - x^n \to -\infty \quad \text{as } x \to \infty$$

$$\text{since} \quad \frac{\ln x}{x^n} \to \frac{1/x}{nx^{n-1}} \to x^{-n} \to 0 \quad \text{and thus} \quad y \to 0.$$

Substituting Eq. (12) into Eq. (10) gives Eq. (1a).