Griffiths Form of the Potential for a Semi-Infinite Slot—C.E. Mungan, Fall 2008

Prove that

$$S \equiv \sum_{m=0}^{\infty} \frac{\exp\left[-(2m+1)\pi x/b\right]\sin\left[(2m+1)\pi y/b\right]}{2m+1} = \frac{1}{2}\tan^{-1}\left[\frac{\sin(\pi y/b)}{\sinh(\pi x/b)}\right]. \tag{1}$$

Using the imaginary part of the Euler identity to rewrite the sine function in the middle expression, we get

$$S = \mathrm{Im}\sum_{m=0}^{\infty} \frac{\exp\left[-(2m+1)\pi x/b\right]\exp\left[i(2m+1)\pi y/b\right]}{2m+1}. \tag{2}$$

Defining the complex variable

$$z \equiv e^{-\pi x/b}e^{i\pi y/b}, \tag{3}$$

we can rewrite Eq. (2) as

$$S = \mathrm{Im}\sum_{m=0}^{\infty} \frac{z^{2m+1}}{2m+1}. \tag{4}$$

However, a standard Maclaurin series is

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots. \tag{5}$$

If we replace z by $-z$ in this series we get

$$\ln(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} - \cdots. \tag{6}$$

Subtracting Eqs. (5) and (6) and dividing by 2 results in

$$\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) = z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots = \sum_{m=0}^{\infty} \frac{z^{2m+1}}{2m+1} \tag{7}$$

so that

$$S = \frac{1}{2}\mathrm{Im}\left[\ln\left(\frac{1+z}{1-z}\right)\right] \tag{8}$$

from Eq. (4). Next we clear the imaginary part out of the denominator of the argument of the logarithm by multiplying and dividing that argument by the complex conjugate of the denominator,

$$\frac{1+z}{1-z} = \frac{1+e^{-\pi x/b}e^{i\pi y/b}}{1-e^{-\pi x/b}e^{i\pi y/b}} \cdot \frac{1-e^{-\pi x/b}e^{-i\pi y/b}}{1-e^{-\pi x/b}e^{-i\pi y/b}} = \frac{1+e^{-\pi x/b}2i\sin(\pi y/b)-e^{-2\pi x/b}}{1+e^{-\pi x/b}2\cos(\pi y/b)+e^{-2\pi x/b}} \tag{9}$$

after multiplying out the terms in the numerator and using the Euler identity, and likewise for the terms in the denominator. Now multiply and divide Eq. (9) by $\exp(\pi x / b)$ and use the definitions of the hyperbolic functions to get

$$\frac{1+z}{1-z} = \frac{\sinh(\pi x / b) + i\sin(\pi y / b)}{\cosh(\pi x / b) - \cos(\pi y / b)} \equiv re^{i\phi} \tag{10}$$

after canceling a factor of 2 out of every term. In the last step, I rewrote this expression in polar form, where the phase angle ϕ is the inverse tangent of the ratio of the imaginary and real parts of the middle expression in Eq. (10),

$$\phi = \tan^{-1}\left[\frac{\sin(\pi y / b)}{\sinh(\pi x / b)}\right]. \tag{11}$$

The desired result now follows from the fact that

$$\ln\left(re^{i\phi}\right) = \ln(r) + i\phi \quad \Rightarrow \quad \mathrm{Im}\left[\ln\left(re^{i\phi}\right)\right] = \phi \tag{12}$$

so that S is equal to half of ϕ according to Eq (8).