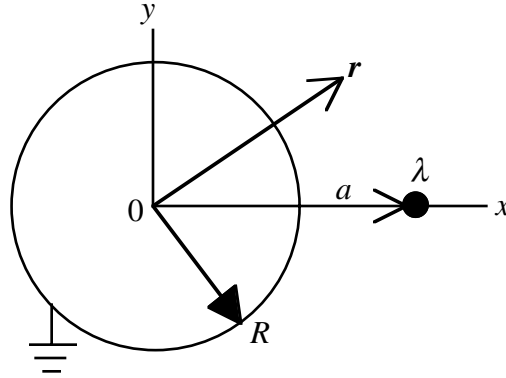


Uniformly Charged Wire Outside a Grounded Cylinder—C.E. Mungan, Spring 1998

Suppose that an infinite wire carrying a uniform linear charge density λ runs parallel to an infinitely long, grounded, conducting cylinder of radius R . The wire is located a distance a away from the center of the cylinder, where $a > R$. The problem is to find the potential everywhere outside the cylinder.



We first find the potential ψ_τ due to the wire alone. Gauss' law applied to a cylindrical gaussian surface of radius r and length L concentric with a wire crossing the origin implies

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{q_{enc}}{\epsilon_0} \Rightarrow E 2\pi r L = \frac{\lambda L}{\epsilon_0} \Rightarrow E = \frac{\lambda \hat{r}}{2\pi\epsilon_0 r}$$

$$\therefore V = -\int \mathbf{E} \cdot d\mathbf{r} = -\frac{\lambda}{2\pi\epsilon_0} \ln r$$

in cylindrical coordinates, where for simplicity the arbitrary constant (which sets the zero of potential) has been chosen to be zero. Shifting the origin a distance a off the wire now gives

$$\begin{aligned} \psi_\tau &= -\frac{\lambda}{2\pi\epsilon_0} \ln|r - a\hat{x}| = -\frac{\lambda}{2\pi\epsilon_0} \ln\sqrt{(x-a)^2 + y^2} \\ &= -\frac{\lambda}{4\pi\epsilon_0} \ln(r^2 - 2ar\cos\theta + a^2). \end{aligned}$$

Note that, unlike for charge distributions of finite size, this potential does not tend to zero at infinity!

Next, the potential ψ_S due to the cylinder alone is a solution of Laplace's equation. In cylindrical coordinates the potential is obviously independent of z and hence is a solution of Laplace's equation in plane polar coordinates. By Boas Problem 13.5.12, this can be written most generally as

$$\psi_S = B_0(C_0 + D_0 \ln r) + \sum_{n=1}^{\infty} \left(C_n r^n + \frac{D_n}{r^n} \right) (A_n \sin n\theta + B_n \cos n\theta).$$

Clearly at large distances from the cylinder, it looks like a line charge and hence $\psi_S \sim \ln r$ as $r \rightarrow \infty$; therefore $C_n = 0$. Absorbing the extraneous constant B_0 into C_0 and D_0 , and similarly D_n into A_n and B_n , now leaves

$$\psi_S = C_0 + D_0 \ln r + \sum_{n=1}^{\infty} \frac{1}{r^n} (A_n \sin n\theta + B_n \cos n\theta).$$

Assembling the solution $\psi = \psi_\tau + \psi_S$, the boundary condition $\psi(R, \theta, z) = 0$ leads to

$$\begin{aligned} \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) &= \frac{\lambda}{4\pi\epsilon_0} \ln(R^2 - 2aR \cos\theta + a^2) \\ &= \frac{\lambda}{4\pi\epsilon_0} \ln a^2 (1 - 2k \cos\theta + k^2) \quad \text{where } k \equiv \frac{R}{a} < 1 \\ &= \frac{\lambda}{2\pi\epsilon_0} \ln a - \frac{\lambda}{2\pi\epsilon_0} \sum_{n=1}^{\infty} \frac{k^n}{n} \cos n\theta \end{aligned}$$

where Schaum's Eq. 23.22 was used in the last step to find the required Fourier series, so that

$$\frac{1}{2}a_0 \equiv C_0 + D_0 \ln R = \frac{\lambda}{2\pi\epsilon_0} \ln a,$$

$$b_n \equiv A_n R^{-n} = 0, \quad \text{and}$$

$$a_n \equiv B_n R^{-n} = -\frac{\lambda}{2\pi\epsilon_0} \frac{1}{n} \left(\frac{R}{a}\right)^n.$$

The fact that a pure cosine series was obtained makes sense in light of the symmetry about the x -axis in this problem. There remains a degree of freedom about how the value of $a_0/2$ is split between C_0 and D_0 . A particularly nice choice is to require ψ to be finite at infinity. Noting that

$$\psi_\tau \rightarrow -\frac{\lambda}{2\pi\epsilon_0} \ln r \quad \text{and} \quad \psi_S \rightarrow D_0 \ln r \quad \text{as } r \rightarrow \infty,$$

we see that we can "cancel" these infinities by putting

$$D_0 = \frac{\lambda}{2\pi\epsilon_0} \Rightarrow C_0 = \frac{\lambda}{2\pi\epsilon_0} (\ln a - \ln R).$$

Thus

$$\begin{aligned} \psi_S &= \frac{\lambda}{2\pi\epsilon_0} \ln \frac{a}{R} + \frac{\lambda}{2\pi\epsilon_0} \ln r - \frac{\lambda}{2\pi\epsilon_0} \sum_{n=1}^{\infty} \frac{\tilde{k}^n}{n} \cos n\theta \quad \text{where } \tilde{k} \equiv \frac{R^2}{ar} < 1 \\ &= \frac{\lambda}{2\pi\epsilon_0} \ln \frac{a}{R} + \frac{\lambda}{2\pi\epsilon_0} \ln r + \frac{\lambda}{4\pi\epsilon_0} \ln(1 - 2\tilde{k} \cos\theta + \tilde{k}^2) \quad \text{using Eq. 23.22 again} \\ &= \frac{\lambda}{2\pi\epsilon_0} \ln \frac{a}{R} + \frac{\lambda}{4\pi\epsilon_0} \ln(r^2 - 2\tilde{a}r \cos\theta + \tilde{a}^2) \quad \text{where } \tilde{a} \equiv \frac{R^2}{a} \end{aligned}$$

and so the final solution is

$$\boxed{\psi(r, \theta, z) = -\frac{\lambda}{4\pi\epsilon_0} \ln(r^2 - 2ar \cos\theta + a^2) + \frac{\lambda}{2\pi\epsilon_0} \ln \frac{a}{R} + \frac{\lambda}{4\pi\epsilon_0} \ln(r^2 - 2\tilde{a}r \cos\theta + \tilde{a}^2)}$$

for the potential everywhere outside the cylinder. We see that this solution is equivalent to an image line charge of linear density $-\lambda$ located at a distance $\tilde{a} < R$ from the origin along the x -axis inside the cylinder. The remaining term $\lambda \ln(a/R)/2\pi\epsilon_0$ is then the potential at infinity. By varying this constant, we can in fact adjust the potential on the conducting cylinder to be any other constant value we wish instead of zero.

Problems

(i) The image method begins by assuming a solution of the form

$$\psi(r, \theta, z) = -\frac{\lambda}{4\pi\epsilon_0} \ln(r^2 - 2ar \cos \theta + a^2) + \frac{\lambda}{4\pi\epsilon_0} \ln(r^2 - 2\tilde{a}r \cos \theta + \tilde{a}^2) + c$$

where \tilde{a} and c are unknown constants. Determine their values by solving the two equations

$$\psi(R, 0, z) = 0 \quad \text{and} \quad \psi(R, \pi, z) = 0$$

for these two unknowns, to verify the above solution.

(ii) Now explicitly verify that $\psi(R, \theta, z) = 0$ for all θ , not just the two values chosen in Problem

(i). This is a matter of simple algebraic manipulation.

(iii) Find the surface charge density $\sigma(\theta)$ on the cylinder by taking the negative gradient of $\psi(r, \theta, z)$ to get the electric field, evaluating this field at the surface of the cylinder, and multiplying the radial component thereof by ϵ_0 , as you learned in E&M I. Hint: The field at the surface of the cylinder has to turn out to be purely radial because the conductor is an equipotential. Use this as a check on your math!

(iv) For extra credit, multiply this surface charge density by $Rd\theta$ and integrate (using tables) around the circle to verify that you get the expected image linear charge density, $-\lambda$.