Hamiltonian Formulation of Geometric Optics—C.E. Mungan, Fall 2005

Because the concepts of kinetic and potential energy are ill-defined for light, there is not an obviously correct Hamiltonian for photons. Rather, one must construct a set of analogies between mechanical and optical quantities. This can be in several different ways, giving rise to different Lagrangian and Hamiltonian formulations of optics, with distinct advantages and disadvantages. In this paper, I will review an elegant treatment presented in D. Drosdoff and A. Widom, Am. J. Phys. **73**, 973 (2005).

We start by considering a ray in a homogenous medium 1 incident at angle θ_1 (relative to the normal) onto the plane interface with a second homogeneous medium 2 into which it refracts at angle θ_2. Since the system has translational invariance in the directions tangential to the plane, the components of the photon's momentum *p* parallel to the plane must be conserved,

$$p_1 \sin\theta_1 = p_2 \sin\theta_2 \,. \tag{1}$$

Now one might guess that since in mechanics, momentum is proportional to speed, $p \propto \upsilon$, and since in optics, speed is inversely proportional to index of refraction, $\upsilon = c / n$, that it should follow that $n \propto 1/p$. However, if we substitute this into Eq. (1), we get an immediate contradiction with Snell's law

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \,. \tag{2}$$

The fix is not to be sought by introducing the photon's relativistic mass *m* in the first proportionality, because $m = E / c^2$ and the photon's energy $E = hf$ is a constant across the interface, since its frequency *f* is fixed by the source independent of the medium.

Instead, Drosdoff and Widom proceed by distinguishing the phase velocity $\upsilon_{\text{phase}} \equiv c / n = \omega / k$ and group velocity $\upsilon_{\text{group}} = d\omega / dk$ for light. Rewriting the above expression for the photon energy in terms of the angular frequency, $E = \hbar\omega$, and noting that the photon momentum is related to the wave vector by $\mathbf{p} = \hbar\mathbf{k}$, we see that

$$n = \frac{ck}{\omega} = \frac{cp}{E} \,. \tag{3}$$

This reduces to the familiar expression $E = cp$ in vacuum. But in a medium, note that $n \propto p$ so that Eqs. (1) and (2) now accord with one another.

The velocity of light $\boldsymbol{\upsilon}$ is not to be identified as the phase velocity in a dispersive medium, however, but rather in general as the group velocity,

$$\upsilon_i \equiv \upsilon_{\text{group},i} = \frac{dE}{dp_i} \tag{4}$$

where $i = \{1,2,3\}$ corresponds to $\{x, y, z\}$ in the usual fashion. Dispersion implies that the refractive index *n* is a function of frequency and thus of energy *E*. In addition, the index is a function of position **r** as we pass across the interface discussed above, or more generally if the light is traveling in an inhomogeneous medium. Thus, we can rewrite Eq. (3) as

$$E = \frac{c\sqrt{p_x^2 + p_y^2 + p_z^2}}{n(\mathbf{r}, E)}, \tag{5}$$

which is an implicit equation for the energy, which we identify as the Hamiltonian *H* (since the coordinates are natural). Hamilton's equations for the motion of the photons then become

$$\dot{r}_i \equiv \upsilon_i = \frac{\partial H(\mathbf{r},\mathbf{p})}{\partial p_i} \tag{6}$$

and

$$\dot{p}_i \equiv F_i = -\frac{\partial H(\mathbf{r},\mathbf{p})}{\partial r_i} \tag{7}$$

for the force **F** on a photon.

Let's see what each of these equations predicts in turn. The differentiation in Eq. (6) is to be performed with **r** held constant. We need to differentiate p_i both explicitly in the numerator and implicitly through $E = H(\mathbf{r},\mathbf{p})$ in the denominator of Eq. (5), so that we get

$$\upsilon_i = \frac{\partial H}{\partial p_i} = \frac{cp_i}{np} - \frac{cp}{n^2}\frac{\partial n}{\partial E}\frac{\partial H}{\partial p_i} \tag{8}$$

which can be rearranged as

$$\frac{\partial H}{\partial p_i}\left[1 + \frac{cp}{n^2}\frac{\partial n}{\partial E}\right] = \frac{cp_i}{np} \quad \Rightarrow \quad \upsilon_i = \frac{cp_i / p}{n + \dfrac{cp}{n}\dfrac{\partial n}{\partial E}}. \tag{9}$$

But

$$\boldsymbol{\upsilon}_{\text{phase}} = \frac{c}{n}\hat{\mathbf{p}} = \frac{c\mathbf{p}}{np}, \tag{10}$$

so that Eq. (9) becomes

$$\boldsymbol{\upsilon} = \frac{n\boldsymbol{\upsilon}_{\text{phase}}}{n + E\,\partial n / \partial E} \quad \text{or} \quad \boldsymbol{\upsilon}_{\text{group}} = \frac{n}{n_{\text{group}}}\boldsymbol{\upsilon}_{\text{phase}} \tag{11}$$

where

$$n_{\text{group}} = n + \omega \frac{\partial n}{\partial \omega} = \frac{n}{1 - \frac{k}{n}\frac{\partial n}{\partial k}} \tag{12}$$

is called the group index of refraction. The second equality in Eq. (12) follows from

$$\omega \frac{\partial n}{\partial \omega} = \omega \frac{\partial n}{\partial k}\frac{\partial k}{\partial \omega} = \omega \frac{\partial n}{\partial k}\frac{n_{\text{group}}}{c} = \frac{k}{n}\frac{\partial n}{\partial k} n_{\text{group}} \tag{13}$$

since $n\upsilon_{\text{phase}} = c = n_{\text{group}}\upsilon_{\text{group}}$ according to the second equality in Eq. (11).

We can evaluate Eq. (7) in a similar fashion,

$$F_i = -\left.\frac{\partial H}{\partial r_i}\right|_{\mathbf{p}} = \frac{cp}{n^2}\left[\frac{\partial n(\mathbf{r},E)}{\partial r_i}\right]_{\mathbf{p}} = \frac{cp}{n^2}\left[\left.\frac{\partial n}{\partial r_i}\right|_E + \left.\frac{\partial n}{\partial E}\right|_{\mathbf{r}}\left.\frac{\partial H}{\partial r_i}\right|_{\mathbf{p}}\right] \tag{14}$$

where for clarity I have explicitly subscripted the variable that is to be held constant during each partial differentiation. I will now drop these needless subscripts and rearrange to obtain

$$-\frac{\partial H}{\partial r_i}\left[1 + \frac{cp}{n^2}\frac{\partial n}{\partial E}\right] = \frac{cp}{n^2}\frac{\partial n}{\partial r_i} \quad \Rightarrow \quad F_i = \frac{E\,\partial n / \partial r_i}{n + E\,\partial n / \partial E} \tag{15}$$

or

$$\mathbf{F} = \frac{\hbar\omega}{n_{\text{group}}}\nabla n\,. \tag{16}$$

This compactly gives the impulsive photon force normal to the plane of discontinuity between two media, as in the Snell's law example at the beginning of this paper.

As I mentioned in the introductory paragraph, there are several other Hamiltonian formulations of geometrical optics. As an example, one can define a Lagrangian $L = n\,dr / ds$ (where s is the arclength along the optical path) starting from Fermat's principle, and then compute the canonical momentum and perform a Legendre transformation to deduce that $H = 0$! Choosing the "optical mass" to be unity and replacing time by arclength, one can then find expressions for the "optical kinetic and potential energies" in terms of the refractive index. See J. Evans and M. Rosenquist, Am. J. Phys. **54**, 876 (1986) and B.N. Turner, JURP **10**, 23 (1991).

If instead we perform a Legendre transformation on the Hamiltonian in the present paper, we find

$$L = \mathbf{p}\bullet\boldsymbol{\upsilon} - H = p\frac{c}{n_{\text{group}}} - E = -E\left(1 - n / n_{\text{group}}\right) \tag{17}$$

which is zero in the absence of dispersion.