Highway Mirages—C.E. Mungan, Summer 2014

The air above a sun-baked road gets heated and the gas expands, so that the density of the air is decreased near the highway. As a result, the index of refraction n of the air is smallest just above the surface and it increases with increasing height z. Let θ indicate the angle that a ray of light from the blue sky makes with the vertical as it descends toward the road. Snell's law states that

$$n\sin\theta = \text{constant} = n_{\min} \tag{1}$$

where $n_{\min}$ is the smallest possible index that the ray will encounter along its trajectory, when the light is traveling parallel to the road so that $\theta = 90°$. At that point, the ray will be at its minimum height above the highway; for simplicity, let us define that height to be $z = 0$. Defining the x axis to be the horizontally forward direction of propagation of the ray, it follows that

$$\sin\theta = \frac{dx}{\sqrt{dx^2 + dz^2}} = \frac{1}{\sqrt{1+(dz/dx)^2}}\,, \tag{2}$$

as can alternatively be derived from the identity $\csc^2\theta = 1 + \cot^2\theta$. Substitute Eq. (2) into (1) and rearrange to obtain

$$\frac{dx}{dz} = \frac{\pm 1}{\sqrt{(n/n_{\min})^2 - 1}}\,. \tag{3}$$

I manually inserted two possible signs into this equation, to allow for the possibility of either a downward-going ray (corresponding to a negative slope) or an upward-traveling ray (corresponding to the positive sign). Given a known functional dependence of the index with height, $n(z)$, Eq. (3) is a differential equation for the trajectory, $x(z)$, of a ray.

Introductory physics textbooks often imply that a decrease in index with decreasing altitude is enough to explain a highway mirage. The idea is that light rays from the sky traveling diagonally downward will continuously refract until they are headed diagonally upward. Those upward-traveling rays will appear as a shimmery bluish patch that forms a water mirage on the road ahead. However, Snell's law only predicts that the rays will become more and more nearly horizontal. Will a monotonic decrease in n with decreasing z always result in a trajectory that curves past the horizontal and actually begin to climb back upward? (Assume the initial slope of the trajectory is downward, but as near 90° as you like, to ensure that the ray does not hit the road before it reaches the height $z = 0$.) In fact it does not, as one can demonstrate by example. Specifically I will consider two cases, one in which the ray does turn around (which would give a mirage) and a second in which the ray asymptotically approaches closer and closer to 90° but never actually gets there, much less turn "beyond" it to start heading back upward.

Case 1: Linear variation in index with height

Let the refractive index be given by

$$n = n_{\min}(1+kz) \tag{4}$$

chosen so that $n = n_{\min}$ at $z = 0$, where k is some parameter with units of reciprocal meters that is approximately proportional to the temperature gradient of the air. If we choose the constant of integration in Eq. (3) such that the ray reaches $z = 0$ at $x = 0$, then the solution is

$$z = \frac{\cosh(kx)-1}{k} \tag{5}$$

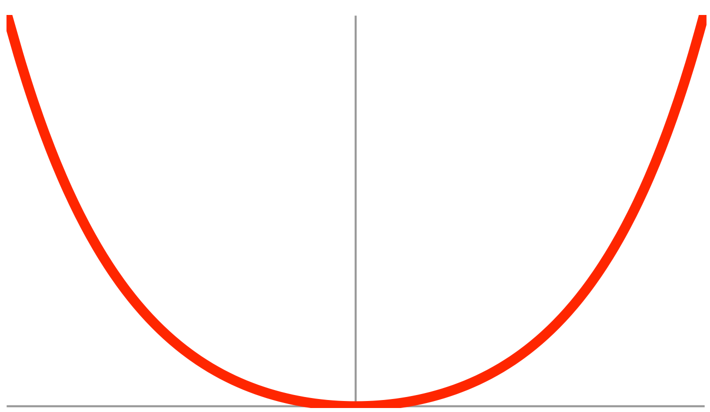
because when it is substituted into Eq. (4) one obtains

$$\left(\frac{n}{n_{\min}}\right)^2 = \cosh^2 kx = 1+\sinh^2 kx \tag{6}$$

which agrees with Eq. (3) written in the form

$$\left(\frac{dz}{dx}\right)^2 = \left(\frac{n}{n_{\min}}\right)^2 - 1 \tag{7}$$

when Eq. (5) is substituted into its left-hand side. Equation (5) is graphed below.



The ray comes in from the upper left and turns around vertically. Notice that both signs of the slope in Eq. (3) arise at any given value of *z*, except at $z = 0$ where total internal reflection occurs.

Case 2: A more complicated variation in index with height

This time suppose the index varies as

$$n = n_{\min}\sqrt{1+(kz)^2} \tag{8}$$

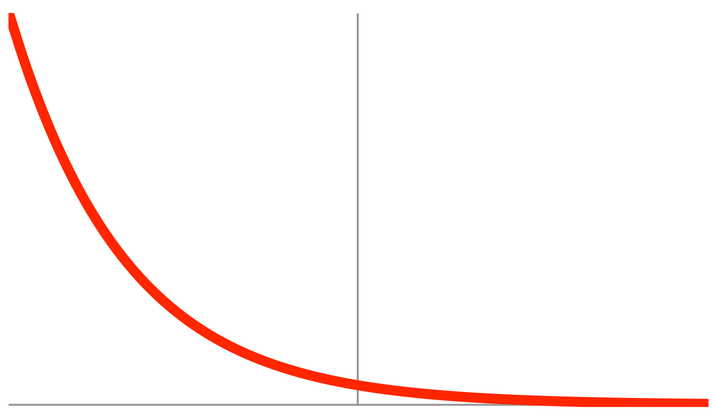
with the same comments about $n_{\min}$ and k as for Eq. (4). Substituting Eq. (8) into (3) leads to

$$\frac{dz}{dx} = \pm kz\,. \tag{9}$$

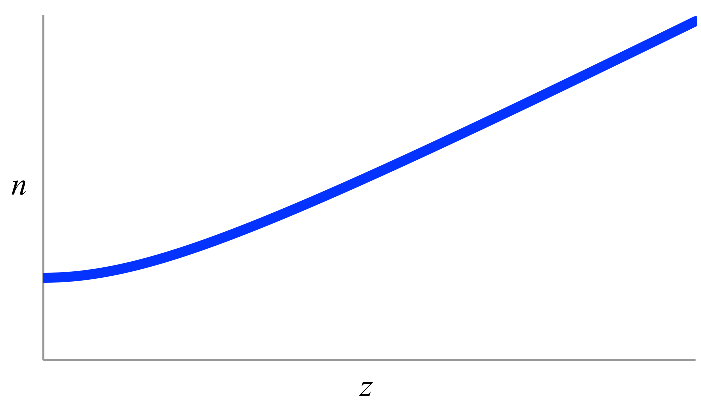
Choosing the minus sign for an initially downward-traveling ray, this equation integrates to

$$z = z_0 e^{-kx} \tag{10}$$

where z_0 is the height of the ray at $x = 0$. This trajectory is graphed below.



The ray never turns around! We can understand this behavior by plotting Eq. (8) below.



The slope approaches zero as $z \to 0$. That is, the index becomes constant and thus the refraction asymptotically tapers off at low altitudes. On the other hand, for large values of z, the slope becomes constant and thus the index variation becomes linear and the far left-hand portion of the red curve above resembles the far left-hand portion of the red curve in the first case.

Thanks to Craig Wiegert for the insights presented here.