**Volume of a Hypersphere—C.E. Mungan, Spring 2010**

Problem: Find the volume $V_n$ of an $n$-dimensional hypersphere of radius $R$. The three lowest values of $n$ are well known. In one dimension, we have a line segment extending a distance $R$ in each direction, so that its length is $V_1 = 2R$. The case of $n = 2$ corresponds to a circle, whose area is $V_2 = \pi R^2$. Finally, $n = 3$ corresponds to a sphere of volume $V_3 = 4\pi R^3 / 3$. Derive a compact formula for the general case.

Method #1: (Courtesy of Bob Sciamanda.) We can write the answer as $V_n(R) = R^n \upsilon_n$, where $\upsilon_n \equiv V_n(1)$ is the volume of a hypersphere of unit radius, since $R$ is the only quantity in the problem with dimensions of length. The volume of any closed solid is

$$V_3 = \int_{z_i}^{z_f} A(z)dz \tag{1}$$

where $A(z)$ is the cross-sectional area of a slab of thickness $dz$ cut through the solid like a loaf of bread, and we integrate from $z_i$ to $z_f$ along any arbitrary axis $z$. It is convenient to choose $z = R\cos\theta$ to be the polar axis, where we integrate upward from $\theta_i = \pi$ to $\theta_f = 0$, and where the area of a slice through the sphere is $V_2(r)$ with $r = R\sin\theta$. Making these substitutions in Eq. (1) gives

$$V_3(R) = R^3 V_2(1)\int_0^{\pi} \sin^3\theta \, d\theta \,. \tag{2}$$

The reader is invited to perform this integral (by using the identity $\sin^2\theta = 1 - \cos^2\theta$) and check that it correctly gives 4/3, as implied in the problem statement above. Generalizing Eq. (2) to $n$ dimensions gives

$$\upsilon_n = \upsilon_{n-1}\int_0^{\pi} \sin^n\theta \, d\theta \,. \tag{3}$$

This equation will correctly reproduce the one-dimensional value $\upsilon_1 = 2$ if we identify $\upsilon_0 \equiv 1$. (That identity can be interpreted as stating that a zero-dimensional sphere is composed of one point.) The problem is now reduced to performing this definite integral and then finding a nonrecursive formula for $\upsilon_n$. The integral is recognized as a beta function, which can be recast in terms of gamma functions as

$$I_n \equiv \int_0^{\pi} \sin^n\theta \, d\theta = B\left(\frac{n+1}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \,. \tag{4}$$

Recall that the gamma function is a generalization of the factorial function, $\Gamma(n+1) = n\Gamma(n)$, where $\Gamma(1)=1$ and $\Gamma(1/2)=\sqrt{\pi}$. Now notice that

$$I_n \cdot I_{n-1} = \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} \cdot \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} = \frac{2\pi}{n}. \tag{5}$$

Substituting this result into Eq. (3) results in a near-miraculous simplification,

$$\begin{aligned} \upsilon_n &= \upsilon_{n-1} I_n = \upsilon_{n-2} I_n I_{n-1} = \upsilon_{n-2}\frac{2\pi}{n} \\ &= \begin{cases} \frac{2\pi}{n}\frac{2\pi}{n-2}\cdots\frac{2\pi}{2}1 & \text{if } n \text{ even} \\ \frac{2\pi}{n}\frac{2\pi}{n-2}\cdots\frac{2\pi}{3}2 & \text{if } n \text{ odd} \end{cases} \end{aligned} \tag{6}$$

where I made use of the facts that $\upsilon_0 = 1$ and $\upsilon_1 = 2$ to terminate the even and odd recursions, respectively. Both of these cases can be written in the single compact way,

$$\boxed{\upsilon_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}} \tag{7}$$

which completes the exercise. The first six values are tabulated below. Notice the recurring patterns in which factors of $\pi$ appear. Also note that $\upsilon_n$ is a maximum for $n=5$ and decreases monotonically to zero for larger values of $n$.

| $n$ | $I_n$ | $\Gamma\left(\frac{n}{2}+1\right)$ | $\upsilon_n$ | $V_n$ |
|---|---|---|---|---|
| 0 | $\pi$ | 1 | 1 | 1 |
| 1 | 2 | $\frac{1}{2}\sqrt{\pi}$ | 2 | $2R$ |
| 2 | $\frac{1}{2}\pi$ | 1 | $\pi$ | $\pi R^2$ |
| 3 | $\frac{4}{3}$ | $\frac{3}{4}\sqrt{\pi}$ | $\frac{4}{3}\pi$ | $\frac{4}{3}\pi R^3$ |
| 4 | $\frac{3}{8}\pi$ | 2 | $\frac{1}{2}\pi^2$ | $\frac{1}{2}\pi^2 R^4$ |
| 5 | $\frac{16}{15}$ | $\frac{15}{8}\sqrt{\pi}$ | $\frac{8}{15}\pi^2$ | $\frac{8}{15}\pi^2 R^5$ |
| 6 | $\frac{5}{16}\pi$ | 6 | $\frac{1}{6}\pi^3$ | $\frac{1}{6}\pi^3 R^6$ |

The surface area of a unit hypersphere, $a_n$, can be obtained by multiplying Eq. (7) by the derivative with respect to $R$ of $R^n$ and then setting $R$ to 1. In other words, multiply Eq. (7) by $n$ to get

$$a_n = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} . \tag{8}$$

Again this area has a maximum, namely at $n = 7$ , and thereafter it decreases to zero.

Method #2: (Courtesy of Matt Springer.) The differential of the volume of a hypersphere of radius $r \equiv R$ is

$$V_n = \upsilon_n r^n \quad \Rightarrow \quad dV_n = \upsilon_n \cdot n r^{n-1} dr . \tag{9}$$

But the usual integral of a Gaussian is

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi} . \tag{10}$$

Multiply this integral by itself *n* times, subscripting each dummy variable *x* by a different index *i* so that we can keep track of them,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \exp\left(-\sum_{i=1}^{n} x_i^2\right)\prod_{i=1}^{n} dx_i = \pi^{n/2} . \tag{11}$$

However, the summation is simply equal to $r^2$ in *n* dimensions, and the product of differentials is just the *n*-dimensional volume element. Writing the volume element in spherical coordinates and performing all of the angular integrals, that element becomes Eq. (9), so that Eq. (11) reduces to

$$\int_0^{\infty} e^{-r^2} \upsilon_n n r^{n-1}\, dr = \pi^{n/2} . \tag{12}$$

Pull the constants $\upsilon_n$ and *n* out of the integral and change variables to $u \equiv r^2$ to get

$$\tfrac{1}{2}\upsilon_n n \int_0^{\infty} e^{-u} u^{n/2-1}\, du = \pi^{n/2} . \tag{13}$$

The remaining integral defines $\Gamma(n/2)$. Finally, noting that

$$\Gamma\left(\frac{n}{2}+1\right) = \frac{n}{2}\Gamma\left(\frac{n}{2}\right), \tag{14}$$

we can rearrange Eq. (13) to get the solution in Eq. (7).