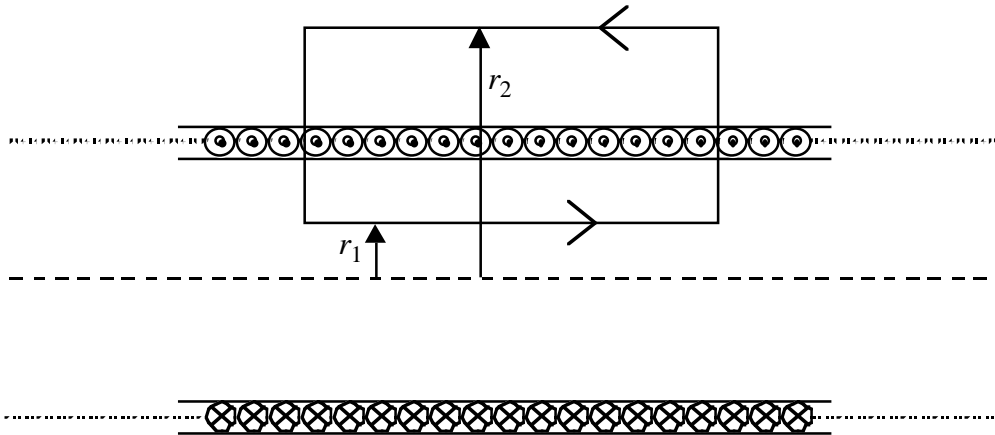


## Magnetic Field Outside an Ideal Solenoid—C.E. Mungan, Spring 2001

It is well known that the longitudinal magnetic field outside an ideal solenoid (i.e., one that is wound infinitely tightly and that is infinitely long) is zero. The question is how to convince a (reasonably bright) student of this of on her first encounter with it. Textbooks typically use Ampère's law for this purpose, as follows.

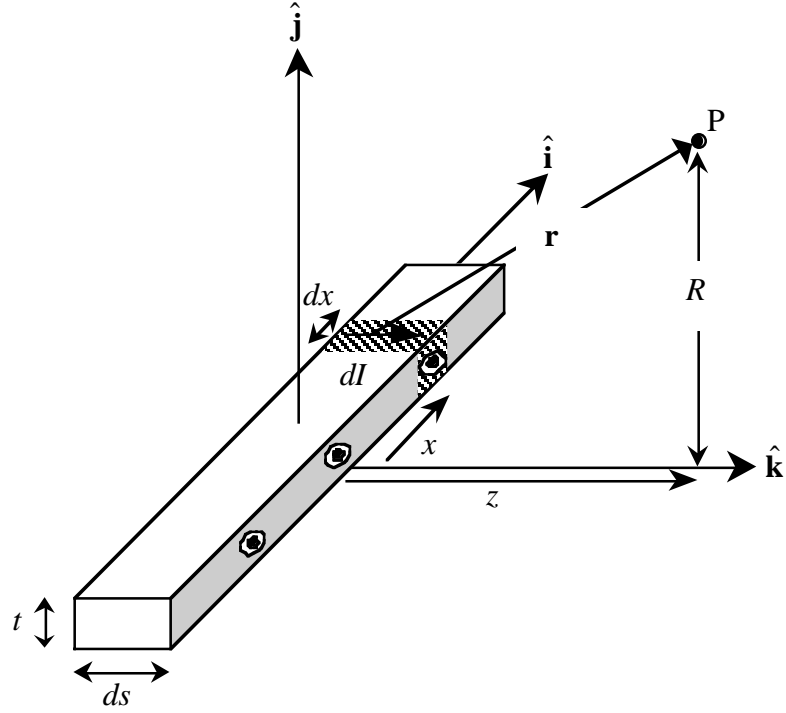


Draw an Ampèrian loop, as sketched above, with two sides parallel to the axis of the solenoid and the other two sides perpendicular to it. Ampère's law applied to this loop gives

$$B(r_2) = \mu_0 nI - B(r_1), \quad (1)$$

where  $n$  is the number of windings per unit length and  $I$  is the current in the solenoid, in the usual manner. Now draw a second Ampèrian loop by increasing  $r_2$  while keeping  $r_1$  fixed. We again get Eq. (1). But the right-hand side of this equation is unchanged. We conclude that  $B(r_2) = \text{constant} \equiv B_{\text{outside}}$ . The argument is now that  $B_{\text{outside}}$  cannot have a nonzero value extending out to infinity and hence it must be zero. This last statement is hard for an introductory student to swallow. After all, the solenoid is infinite in length and hence extends to infinity! Or consider an infinite sheet of uniformly distributed charge—it has a constant electric field all the way out to infinity. As instructors, we usually have to spend some time explaining what this means (why it is reasonable for an ideal infinite sheet and how it corresponds to a real, i.e. finite, sheet of charge). It is disconcerting, to say the least, to get students to buy into this and then a few chapters later try to also get them to accept the above argument for why  $B_{\text{outside}}$  must be zero. In fact, as I will argue here, a better argument instead of saying that you cannot have a nonzero field extending to infinity, is that it happens to be canceled by *another* nonzero field which also extends to infinity!

Specifically, consider any longitudinal strip of the solenoid to be a semi-infinite sheet of current with the following dimensions: infinite width in the longitudinal direction, infinitesimal thickness  $t$  in the radial direction, and differential length  $ds$  in the azimuthal direction. Let's find the magnetic field that this current strip produces at an arbitrary point P in space. For this purpose, it is convenient to switch to a rectangular coordinate system where  $z$  points in the current direction,  $x$  points parallel to the axis of the solenoid, and  $y$  is directed perpendicularly to the solenoid. Choose the origin of the coordinate system to be in the strip and such that P has zero  $x$ -coordinate, as sketched on the top of the next page.



We see from this diagram that

$$d\mathbf{s} \times \mathbf{r} = (dz\hat{\mathbf{k}}) \times (-x, R, z) = -(R\hat{\mathbf{i}} + x\hat{\mathbf{j}}) dz. \quad (2)$$

Furthermore the differential current is  $dI = Jtdx \equiv Kdx$  where  $J$  is the volume current density and  $K$  is the surface current density. Hence the Biot-Savart law becomes

$$\begin{aligned} d\mathbf{B} &= \int \frac{\mu_0}{4\pi} \frac{dI d\mathbf{s} \times \mathbf{r}}{r^3} = -\frac{\mu_0 K}{4\pi} dz \left[ R\hat{\mathbf{i}} \int_{-\infty}^{+\infty} \frac{dx}{(x^2 + z^2 + R^2)^{3/2}} + \hat{\mathbf{j}} \int_{-\infty}^{+\infty} \frac{x dx}{(x^2 + z^2 + R^2)^{3/2}} \right] \\ &= -\frac{\mu_0 K}{4\pi} dz \lim_{x \rightarrow \infty} \left[ R\hat{\mathbf{i}} \frac{x}{(z^2 + R^2)\sqrt{x^2 + z^2 + R^2}} - \hat{\mathbf{j}} \frac{1}{\sqrt{x^2 + z^2 + R^2}} \right]_{-x}^{+x} \\ &= -\frac{\mu_0 K}{4\pi} dz \frac{2R\hat{\mathbf{i}}}{z^2 + R^2}. \end{aligned} \quad (3)$$

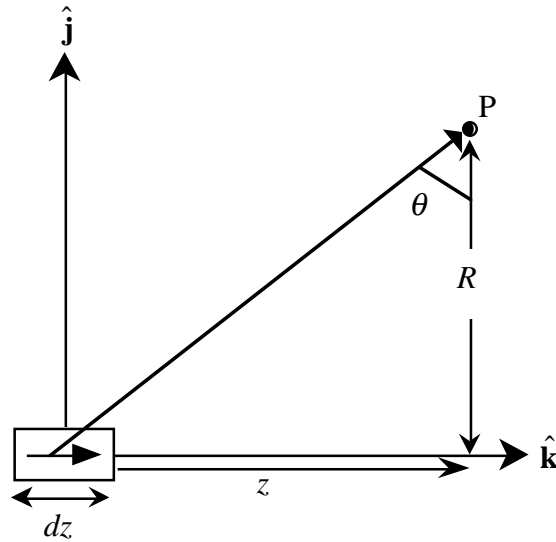
Next, the  $yz$ -plane has been redrawn at the top of the following page. We see from it that

$$z = R \tan \theta \Rightarrow dz = R \sec^2 \theta d\theta = \frac{z^2 + R^2}{R} d\theta \quad (4)$$

and hence we finally obtain the very simple result

$$\frac{dB}{d\theta} = \frac{\mu_0 K}{2\pi}. \quad (5)$$

Note that the right-hand side is a constant—in particular, it is independent of both  $R$  and  $\theta$ !

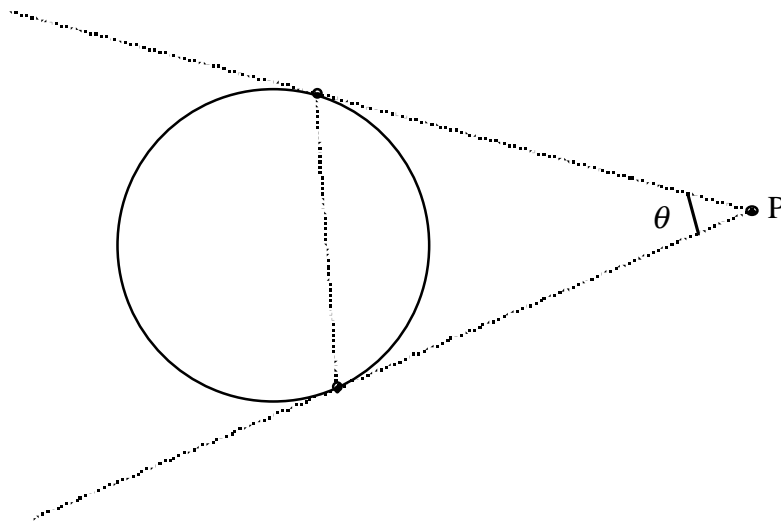


As a check, if we integrate Eq. (5) from  $-\pi/2$  to  $+\pi/2$  we obtain  $B = \mu_0 K / 2$ , which is indeed the magnetic field of an infinite sheet of current.

On the other hand, if we integrate it around a closed curve we get

$$\oint_C d\theta = \begin{cases} 0 & \text{if P is outside } C \Rightarrow B_{\text{outside}} = 0 \\ 2\pi & \text{if P is enclosed by } C \Rightarrow B_{\text{inside}} = \mu_0 nI \end{cases} \quad (6)$$

using the fact that  $K \equiv dI / dx = nI$ . In fact what is happening, as we see in the figure below which shows a cross section of a round solenoid, is that we get  $B = \mu_0 nI \theta / 2\pi$  from the near side and an equal but opposite contribution from the far side.



The solenoid's cross section need not be round—Eq. (6) holds for a right cylinder of *any* shape. Another way to see this is to tile an irregularly shaped cross section by a space-filling set of round disks.