Further Thoughts on the Ideal Solenoid—C.E. Mungan, Spring 2003

In a previous document, I illustrated one method of rigourously deriving the magnetic field inside and outside an ideal (infinitely long and infinitely tightly wound) solenoid of arbitrary cross-sectional shape using the Biot-Savart law alone. But it would take some time to work that method out on the board. Here is another nice approach suggested by Gene Mosca that only uses concepts already developed in a calculus-based introductory course.

1. Integrate the magnetic field of a set of loops on-axis to find that of an ideal round solenoid on-axis:

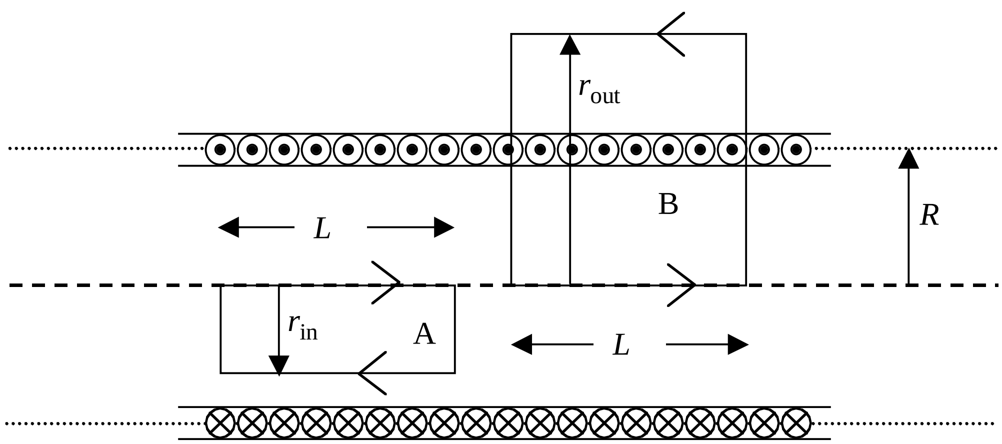
$$B_{\text{loop}} = \frac{\mu_0 R^2 I}{2(x^2+R^2)^{3/2}} \quad \Rightarrow \quad B_{\text{solenoid}}^{\text{on-axis}} = \int_{-\infty}^{+\infty} \frac{\mu_0 R^2 nIdx}{2(x^2+R^2)^{3/2}}. \tag{1}$$

Change variables to $x = R\tan\theta$ to find

$$B_{\text{solenoid}}^{\text{on-axis}} = \frac{\mu_0 nIR^2}{2} \int_{-\infty}^{+\infty} \frac{R\sec^2\theta d\theta}{R^3\sec^3\theta} = \frac{\mu_0 nI}{2}\sin\theta\Big|_{-\pi/2}^{+\pi/2} = \mu_0 nI \tag{2}$$

as expected.

2. Apply Ampère’s law to the two indicated loops below to get to any arbitrary point inside or outside the ideal round solenoid.



For loop A we obtain

$$B_{\text{solenoid}}^{\text{on-axis}} L - B_{\text{solenoid}}^{\text{off-axis}} L = 0 \quad \Rightarrow \quad B_{\text{solenoid}}^{\text{interior}} = \mu_0 nI \tag{3}$$

for any $r_{\text{in}} < R$, while for loop B we get

$$B_{\text{solenoid}}^{\text{on-axis}} L - B_{\text{solenoid}}^{\text{off-axis}} L = \mu_0 NI \quad \Rightarrow \quad B_{\text{solenoid}}^{\text{exterior}} = 0 \tag{4}$$

for all $r_{\text{out}} > R$.

3. Tile any arbitrarily shaped solenoid with round solenoids to establish the final result:

$$\boxed{B_{\text{solenoid}} = \begin{cases} \mu_0 nI & \text{for any interior point} \\ 0 & \text{outside} \end{cases}}. \tag{5}$$

References

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