Image Charges for Two Concentric Grounded Spheres—C.E. Mungan, Spring 2020

First consider the problem of finding the image charge Q_1 at radius R_1 inside a grounded metal sphere of radius R due to a point charge q located at radius r outside the sphere, as sketched in Fig. 1.

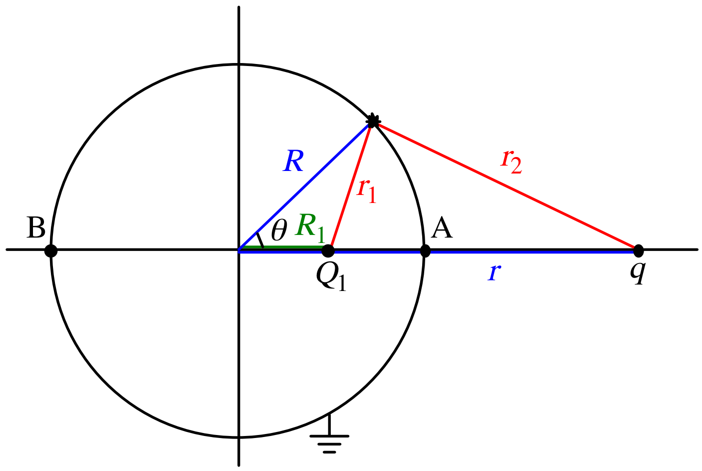

Figure 1

Clearly Q_1 must be off center. (We could add a second image charge at the center of the sphere if we wanted the potential of the sphere to be nonzero.) By symmetry it must be on the line between the center of the sphere and charge q, and it must be of opposite sign as q. The quickest way to find the values of Q_1 and R_1 are to zero out the potential at two points on the surface of the sphere. At point A we must have

$$\frac{kQ_1}{R-R_1}+\frac{kq}{r-R}=0 \tag{1}$$

where k is the Coulomb constant, whereas at point B we require

$$\frac{kQ_1}{R+R_1}+\frac{kq}{r+R}=0\,. \tag{2}$$

Simultaneous solution of Eqs. (1) and (2) leads to

$$R_1=\frac{R^2}{r} \tag{3}$$

and

$$Q_1=-q\frac{R}{r}\,. \tag{4}$$

Note that $R_1<R$ and $|Q_1|<|q|$. To verify that this solution works for all other points on the sphere, consider the arbitrary asterisked point at angle θ in Fig. 1. From the law of cosines we have

$$r_1 = \sqrt{\left(\frac{R^2}{r}\right)^2 + R^2 - 2R\frac{R^2}{r}\cos\theta} \tag{5}$$

using Eq. (3), and likewise

$$r_2 = \sqrt{r^2 + R^2 - 2Rr\cos\theta} = \frac{r}{R}\sqrt{R^2 + \left(\frac{R^2}{r}\right)^2 - 2R\frac{R^2}{r}\cos\theta} = \frac{r}{R}r_1 . \tag{6}$$

Substituting Eqs. (4) and (6) into the following equality verifies that

$$\frac{kQ_1}{r_1} + \frac{kq}{r_2} = 0 \tag{7}$$

as desired.

This idea works equally well for a point charge q at radius r inside a grounded metal sphere of radius R. Adding primes to the left-hand sides of Eqs. (3) and (4) to indicate the charge and position of this external image charge, we thus have

$$R_1' = \frac{R^2}{r} \tag{8}$$

and

$$Q_1' = -q\frac{R}{r} . \tag{9}$$

Observe that $R_1' > R$ and $|Q_1'| > |q|$, in contrast to the two inequalities for the inner image charge given in the line following Eq. (4). We are going to be interested in image charges outside a sphere of radius 3R, in which case Eqs. (8) and (9) become

$$R_1' = \frac{9R^2}{r} \tag{10}$$

and

$$Q_1' = -q\frac{3R}{r} . \tag{11}$$

Specifically, the problem by Seth Rittenhouse is to find the electric field at the asterisked point in Fig. 2.

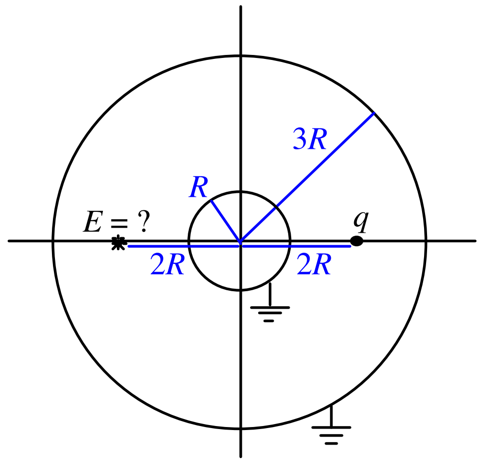

Figure 2

Charge q induces image charge Q_1 at R_1 where we put $r = 2R$ in Eqs. (3) and (4) to get

$$R_1 = \tfrac{1}{2}R \quad \text{and} \quad Q_1 = -\tfrac{1}{2}q\,. \tag{12}$$

In addition, it induces image charge Q_1' at R_1' where we put $r = 2R$ in Eqs. (10) and (11) to get

$$R_1' = \tfrac{9}{2}R \quad \text{and} \quad Q_1' = -\tfrac{3}{2}q\,. \tag{13}$$

However, image charge Q_1' induces image charge Q_2 at R_2 where we put $r = R_1'$ and $q = Q_1'$ in Eqs. (3) and (4) to get

$$R_2 = \tfrac{2}{9}R \quad \text{and} \quad Q_2 = +\tfrac{1}{3}q\,. \tag{14}$$

Likewise, image charge Q_1 induces image charge Q_2' at R_2' where we put $r = R_1$ and $q = Q_1$ in Eqs. (10) and (11) to get

$$R_2' = 18R \quad \text{and} \quad Q_2' = +3q\,. \tag{15}$$

These two image charges then induce the image charges

$$R_3 = \tfrac{1}{18}R \quad \text{and} \quad Q_3 = -\tfrac{1}{6}q\,. \tag{16}$$

and

$$R_3' = \tfrac{81}{2}R \quad \text{and} \quad Q_3' = -\tfrac{9}{2}q\,. \tag{17}$$

It is clear that we are getting an even and an odd infinite series for both the inner and outer image charges, given by

$$R_{2n+1} = \tfrac{1}{2}R\,9^{-n} \quad \text{and} \quad Q_{2n+1} = -\tfrac{1}{2}q\,3^{-n} \tag{18}$$

and

$$R_{2n+2} = \tfrac{2}{9}R\,9^{-n} \quad \text{and} \quad Q_{2n+2} = +\tfrac{1}{3}q\,3^{-n} \tag{19}$$

on the inside (where $n = 0,1,2,\ldots$) and by

$$R'_{2n+1} = \tfrac{9}{2} R 9^n \quad \text{and} \quad Q'_{2n+1} = -\tfrac{3}{2} q 3^n \tag{20}$$

and

$$R'_{2n+2} = 18 R 9^n \quad \text{and} \quad Q'_{2n+2} = +3q 3^n \tag{21}$$

on the outside. Finally the total electric field at the asterisked point in Fig. 2 is the sum of that due to these four infinite series plus that due to the original point charge q,

$$E = \frac{kq}{(4R)^2} + \sum_{n=0}^{\infty}\left[\frac{-k\frac{1}{2}q3^{-n}}{\left(2R+\frac{1}{2}R9^{-n}\right)^2} + \frac{k\frac{1}{3}q3^{-n}}{\left(2R+\frac{2}{9}R9^{-n}\right)^2} + \frac{-k\frac{3}{2}q3^{n}}{\left(2R+\frac{9}{2}R9^{n}\right)^2} + \frac{k3q3^{n}}{\left(2R+18R9^{n}\right)^2}\right] \tag{22}$$

directed leftward. To evaluate this expression as a multiple of kq/R^2, copy and paste the following line into Mathematica,

```
N[1/16+Sum[-(1/2/(3^n*(2+1/2/9^n)^2))+1/3/(3^n*(2+2/9/9^n)^2)-
((3/2)*3^n)/(2+(9/2)*9^n)^2+(3*3^n)/(2+18*9^n)^2,{n,0,Infinity}]]
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to get a result of 0.000 526 476. After pasting it into Mathematica, you can convert this line to "StandardForm" to get it to look more like Eq. (22). It makes sense that this answer is small, slightly less than 1% of $kq/(4R)^2$, because the asterisked point lies in free space between two grounded surfaces 180° away from charge q.

We can calculate a few other things. To find the potential at the asterisked point, simply delete the squares in the five denominators in Eq. (22) to get $V = 0.000\ 602\ 767\ kq/R$. To find the total induced charge on the surface of the inner sphere, sum up all the image charges inside it to obtain

$$Q_{\text{inner}} = \sum_{n=0}^{\infty}\left(\frac{1}{3}-\frac{1}{2}\right)q3^{-n} = -\frac{q}{6}\frac{1}{1-1/3} = -\frac{q}{4}. \tag{23}$$

Since every field line that starts on the $+q$ point charge must end on the two grounded spheres, the total induced charge on the the surface of the outer sphere must be $Q_{\text{outer}} = -3q/4$. If we try to add the image charges outside it, however, we find a divergent series. Suppose we ignore that fact and proceed nevertheless as in Eq. (23). We get

$$Q_{\text{outer}} = \sum_{n=0}^{\infty}\left(3-\frac{3}{2}\right)q3^{n} = \frac{3q}{2}\frac{1}{1-3} = -\frac{3q}{4} \tag{24}$$

which is the right answer! That shows one can do sneaky things with divergent series. But there is another way to get the answer which does not rely on such ruses. We can calculate the potential at the origin in Fig. 2 due to the image charges for the outer sphere. That answer must equal $kQ_{\text{outer}}/(3R)$ since every point on its surface is a distance $3R$ away from the origin. Thus

$$\frac{kQ_{\text{outer}}}{3R} = \sum_{n=0}^{\infty}\left[\frac{-k\frac{3}{2}q3^n}{\frac{9}{2}R9^n} + \frac{k3q3^n}{18R9^n}\right] = \frac{-kq}{6R}\sum_{n=0}^{\infty}3^{-n} = \frac{-kq}{6R}\frac{1}{1-1/3} \tag{25}$$

which simplifies to $Q_{\text{outer}} = -3q/4$. If you try this idea for the inner sphere, you will find that it now gives a divergent series, but if you assume the same sum for 3^n as in Eq. (24), namely $-1/2$, you again get the right answer for Q_{inner}.