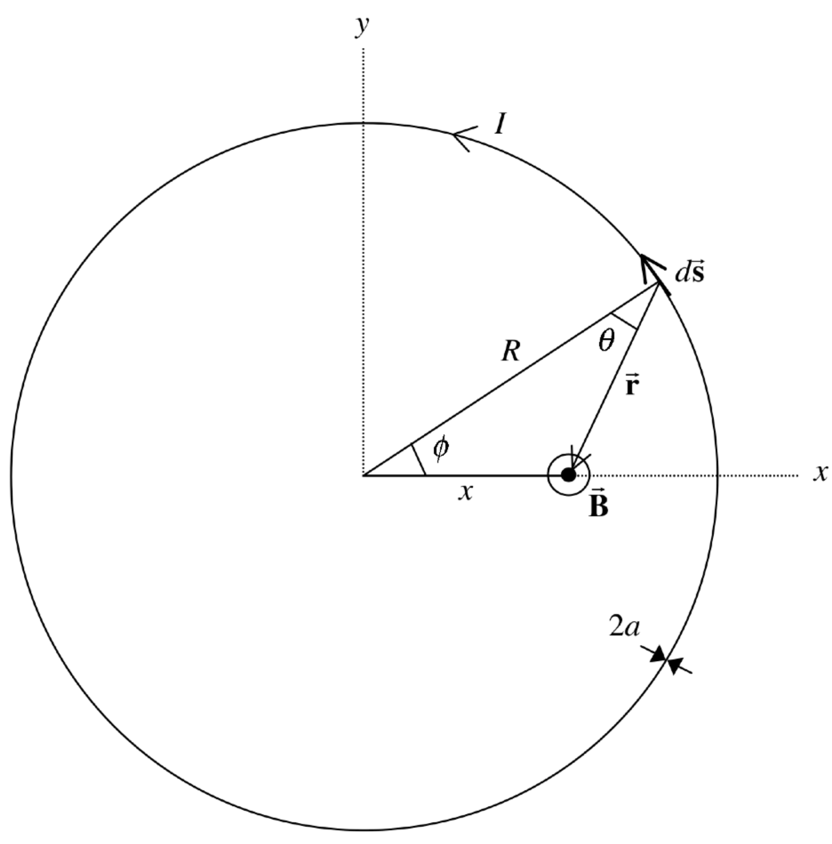
Inductance of a Flat Circular Coil—C.E. Mungan, Spring 2012

The goal of this exercise is to theoretically estimate the self-inductance of PASCO's field coil model EM-6711. It consists of $N = 200$ turns of AWG 22 copper wire (i.e., approximately 0.65 mm in diameter). The coil has an average radius of $R = 10.5$ cm, with an inner radius of $R_{\text{inner}} = 9.8$ cm and an outer radius of $R_{\text{outer}} = 11.05$ cm. If we were to model the 200 strands by a single equivalent thick wire, it would thus have a radius of $a = (R_{\text{outer}} - R_{\text{inner}})/2 = 0.625$ cm.

So consider the magnetic field everywhere in the interior plane of a single circular loop of wire carrying current I, where the loop has radius R and the wire has radius a. Without loss of generality, we put the origin at the center of the loop and let the z-axis point normal to the plane of the loop. We can also orient the x-axis so that the point at which we want to find the magnetic field $\vec{\mathbf{B}}$ is located at coordinates $0 \le x \le R - a$, $y = 0$, and $z = 0$ as sketched below.



The Biot-Savart law is

$$\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2} = \frac{\mu_0 I}{4\pi} \int \frac{(R d\phi)(\cos\theta)\hat{\mathbf{k}}}{r^2}. \tag{1}$$

The law of sines implies

$$\frac{\sin\theta}{x} = \frac{\sin\phi}{r} \quad \Rightarrow \quad \cos\theta = \frac{\sqrt{r^2 - x^2(1-\cos^2\phi)}}{r}\,, \tag{2}$$

while the law of cosines gives

$$r^2 = R^2 + x^2 - 2xR\cos\phi\,. \tag{3}$$

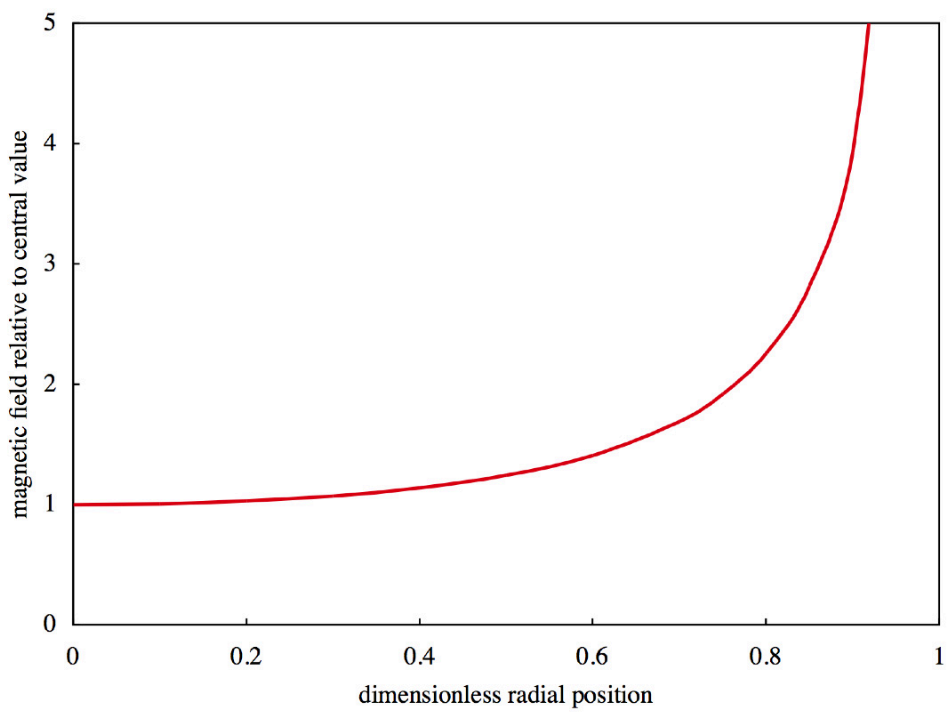
Substitute Eq. (2) into (1) and then replace all occurrences of r with Eq. (3) to get

$$B = \frac{\mu_0 IR}{4\pi}\int \frac{\sqrt{R^2 + x^2\cos^2\phi - 2xR\cos\phi}}{\left(R^2 + x^2 - 2xR\cos\phi\right)^{3/2}}\,d\phi\,. \tag{4}$$

Define the dimensionless radial position $\rho \equiv x / R$ and double the integral over half the circumference of the loop to obtain

$$B(\rho) = \frac{B_0}{\pi}\int_0^{\pi} \frac{1-\rho\cos\phi}{\left(1+\rho^2 - 2\rho\cos\phi\right)^{3/2}}\,d\phi \tag{5}$$

where $B_0 \equiv \mu_0 I / 2R$ is the field at the center of the loop. I computed $B_{\text{rel}} \equiv B / B_0$ versus ρ in intervals of 0.05 from 0 to 0.95 in Mathematica and plotted the results in Excel below. This graph and Eq. (5) agree with the published results in H. Erlichson, AJP **57**, 607 (1989).



The magnetic field rises monotonically starting from the central value B_0, so using that central value to compute the inductance would lead to an underestimate. The field would diverge if x could reach R, but we prevent that by cutting it off when we hit the surface of the wire at $x_{\max} = R - a$.

The inductance of the coil can now be computed as $L \equiv N\Phi / I$ where the magnetic flux linking the coil is

$$\Phi = \int_0^{x_{\max}} NB(x) 2\pi x \, dx \, . \tag{6}$$

The factor of N inside the integral arises from the fact that each winding increases the total magnetic field. Substituting Eq. (5), we obtain

$$L = \mu_0 N^2 \pi R \int_0^{1-a/R} B_{\mathrm{rel}}(\rho) \rho \, d\rho \, . \tag{7}$$

Substituting the numerical values given in the first paragraph, Mathematica obtains L = 14.7 mH. Jim Huddle measured the actual inductance to be 16.7 mH, in fair agreement. Note that the answer depends sensitively on the value of a, explaining the discrepancy. We merely have to reduce a from 6.25 to 4.40 mm to reproduce the experimental result.