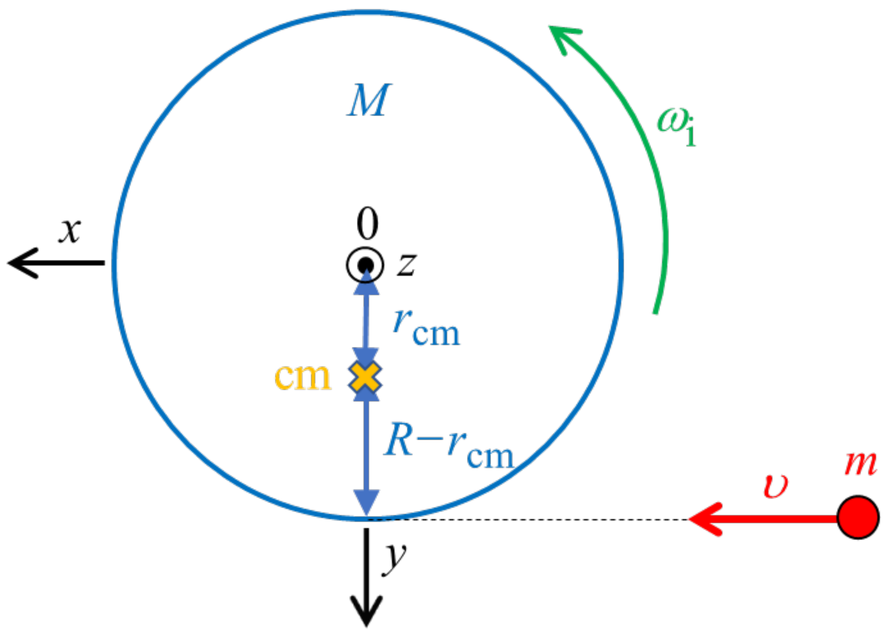
Inelastic Rotational Collision Between Two Free Objects—C.E. Mungan, Fall 2022

A satellite in the form of a uniform right circular cylinder of mass M and radius R is rotating about its central axis of symmetry in the counter-clockwise (ccw) direction at initial angular speed ω_i. As illustrated in the following figure, its bottom rim is struck tangentially (at its mid-length along the z direction) by a point mass m incident from the right at speed υ (relative to the axis of the satellite) and sticks to it. (Alternatively one could imagine a puck spinning in place on an air hockey table that is struck by a wad of chewing gum that slides tangentially into it.) Consequently, the angular speed of the system of satellite and embedded mass decreases to a final value of ω_f. What is υ in terms of the other given quantities?



The position of the center-of-mass (cm) of the system at the moment of impact relative to the axis of the cylinder is

$$r_\text{cm} = \frac{M \cdot 0 + m \cdot R}{m+M} = \frac{mR}{m+M} \quad \Rightarrow \quad R - r_\text{cm} = \frac{MR}{m+M}. \tag{1}$$

Equate the ccw angular momentum of the system about its cm before and after impact. We can calculate the angular momentum of any object as the sum of the spin angular momentum about the cm of that particular object (not the cm of the system) plus the orbital angular momentum of the object's cm relative to an inertial axis (which in the present case we choose to be at the cm of the system that by conservation of linear momentum has constant leftward velocity). We choose our inertial frame of reference to be comoving with the axis 0 of the satellite before impact. Thus the initial angular momentum of the cylinder is simply its spin angular momentum $I_0\omega_\text{i}$; it has no initial orbital angular momentum because its axis 0 is not initially moving in our chosen frame of reference. The moment of inertia of the uniform cylinder about its own cm at 0 is $I_0 = MR^2/2$. A point mass never has spin angular momentum because it has no size. Its orbital angular momentum is clockwise (cw) and thus negative with magnitude $m\upsilon(R - r_\text{cm})$ relative to the cm of the system. After impact, the system has no orbital angular momentum relative to its cm. Its spin orbital angular momentum is the product of the final ccw angular speed ω_f and the sum of the moments of inertia of the cylinder and point mass about the cm of the system. We use the parallel-axis theorem to find the moment of inertia of the cylinder about the cm of the system. Putting it all together, we get

$$\tfrac{1}{2}MR^2\omega_\mathrm{i} - m\upsilon(R - r_\mathrm{cm}) = \left[\tfrac{1}{2}MR^2 + Mr_\mathrm{cm}^2 + m(R - r_\mathrm{cm})^2\right]\omega_\mathrm{f} \,. \tag{2}$$

This relation rearranges into

$$\frac{\upsilon}{R} = \left(1 + \frac{M}{m}\right)\frac{\omega_\mathrm{i}}{2} - \left(3 + \frac{M}{m}\right)\frac{\omega_\mathrm{f}}{2} \tag{3}$$

using Eq. (1). For example if R = 1.3 m, M = 550 kg, m = 29 kg, ω_i = 2.8 rad/s, and ω_f = 1.2 rad/s then υ = 19.2 m/s. In contrast, consider what would happen if the cylinder had an axle through its center that is fixed in linear position like a pottery wheel. Then Eq. (2) would instead be

$$\tfrac{1}{2}MR^2\omega_\mathrm{i} - m\upsilon R = \left[\tfrac{1}{2}MR^2 + mR^2\right]\omega_\mathrm{f} \tag{4}$$

for conservation of angular momentum about the axle. That equation rearranges into

$$\frac{\upsilon}{R} = \left(\frac{M}{m}\right)\frac{\omega_\mathrm{i}}{2} - \left(2 + \frac{M}{m}\right)\frac{\omega_\mathrm{f}}{2} \tag{5}$$

written in a way to facilitate comparison with Eq. (3). This result implies a smaller value than above, namely υ = 18.2 m/s. However, in the limit that $m \ll M$, both Eq. (3) and (5) reduce to

$$\frac{\upsilon}{R} \approx \frac{M}{2m}(\omega_\mathrm{i} - \omega_\mathrm{f}) \tag{6}$$

which agrees with Eq. (4) after neglecting the second term in its square brackets. It makes sense that both systems give the same answer for υ in the limit $m \ll M$ because Eq. (1) then implies that $r_\mathrm{cm} \approx 0$. That explains why Eqs. (3) and (5) give similar results for υ in the numerical example above, when m is only about 5% of M. In contrast, if m were increased to $M(\omega_\mathrm{i} - \omega_\mathrm{f})/(2\omega_\mathrm{f}) = 367$ kg then Eq. (5) would imply υ = 0, while Eq. (3) would give υ = 1.04 m/s. As a check, Eq. (6) correctly leads to $\omega_\mathrm{i} = \omega_\mathrm{f}$ if m = 0, corresponding to no impact.

Readers looking to extend these results may wish to calculate what percentage of the mechanical energy of the system is lost in this inelastic collision. In that case, they will first need to calculate the final translational speed of the cm of the system using conservation of linear momentum to get

$$\upsilon_\mathrm{cm} = \frac{m\upsilon}{m + M} \,. \tag{7}$$

I find that 95.9% of the mechanical energy is lost for the numbers given above.

Appendix: Three other ways to write conservation of angular momentum

<u>reference</u>: EJP **18**, 363 (1997)

Calculate angular momentum of both the cylinder and point mass as the sum of their spin and orbital terms, both immediately before and after impact, for three different choices of axes that are fixed not to the objects but to our inertial frame of reference. (The final angular speed of the cylinder about the cm of the system is equal to that about any other point on the cylinder.) With reference to the figure below, the first choice is an axis that happens to pass through point 0 at the moment of impact. Then we get

$$\left[\tfrac{1}{2}MR^2\omega_{\mathrm{i}}+0\right]+\left[0-m\upsilon R\right]=\left[\tfrac{1}{2}MR^2\omega_{\mathrm{f}}+0\right]+\left[0-m\upsilon_{\mathrm{rim}}R\right]. \tag{8}$$

To be fully explicit about the details, the first bracket contains the sum of the spin + orbital angular momenta for the cylinder initially, the second bracket the same sum for the point mass initially, the third bracket the sum for the cylinder finally, and the fourth bracket the sum for the point mass finally. Here, the final leftward speed of a point on the bottom rim of the cylinder is

$$\upsilon_{\mathrm{rim}}=\upsilon_{\mathrm{cm}}-(R-r_{\mathrm{cm}})\omega_{\mathrm{f}}\,. \tag{9}$$

The second choice is an axis that happens to pass through the bottom rim at the moment of impact, giving

$$\left[\tfrac{1}{2}MR^2\omega_{\mathrm{i}}+0\right]+\left[0+0\right]=\left[\tfrac{1}{2}MR^2\omega_{\mathrm{f}}+M\upsilon_0 R\right]+\left[0+0\right] \tag{10}$$

where the final leftward speed of the center of the cylinder is

$$\upsilon_0=\upsilon_{\mathrm{cm}}+r_{\mathrm{cm}}\omega_{\mathrm{f}}\,. \tag{11}$$

Finally the third choice of axis passes through the cm of the system at the instant of impact, so that

$$\left[\tfrac{1}{2}MR^2\omega_{\mathrm{i}}+0\right]+\left[0-m\upsilon(R-r_{\mathrm{cm}})\right]=\left[\tfrac{1}{2}MR^2\omega_{\mathrm{f}}+M\upsilon_0 r_{\mathrm{cm}}\right]+\left[0-m\upsilon_{\mathrm{rim}}(R-r_{\mathrm{cm}})\right]. \tag{12}$$

Using Eqs. (1), (7), (9), and (11), the reader can show that Eqs. (8), (10), and (12) each rearrange into Eq. (3).