

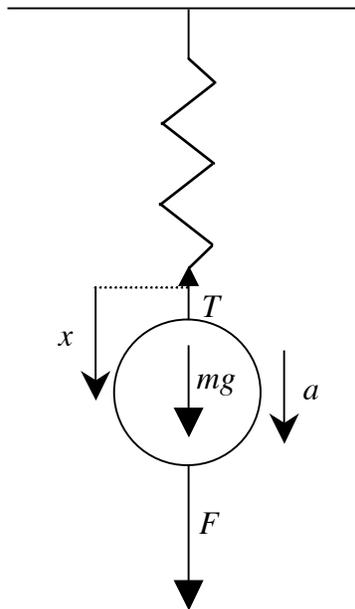
Inertia Ball—C.E. Mungan, Fall 2015

A well-known lecture demonstration consists of hanging a heavy ball from a fixed bar by a thin string. An identical string is connected to a hook on the bottom of the mass. If the lower string is pulled slowly, the upper string eventually breaks, because it supports both the applied force and the hanging weight. However, if the lower string is suddenly jerked, it is the one which breaks. The traditional explanation is that the inertia of the mass keeps it from moving and hence from stretching the upper string. This idea is basically correct but crude. It becomes satisfying only when supplemented by a quantitative analysis. Fortunately a simple analysis accessible to an introductory student is not hard and is a nice exercise in integration.

The upper string can be viewed as a rather stiff but massless spring of force constant k . Let the weight have mass m and a time-dependent downward acceleration a . Denote its equilibrium position as zero before any force is applied to the lower string, and any subsequent downward displacement as x , so that $a = \ddot{x}$. Therefore the tension in the upper string is

$$T = mg + kx . \quad (1)$$

Although the two strings have identical properties, the stretch of the lower string is of no interest and merely affects how the applied force is transmitted to the ball, which depends on how you move your hand in any case. So for simplicity, I will disregard the elasticity of the lower string and assume the applied force F increases linearly from zero up to the point that one of the strings break, with a rate of increase \dot{F} that we can vary from trial to trial. A free-body diagram is thus as follows.



Applying Newton's second law to the ball gives

$$ma = F + mg - T = F - kx \quad (2)$$

using Eq. (1) in the second step. Substituting $F = \dot{F}t$ and $a = \ddot{x}$ gives a second-order ODE, which is linear and inhomogeneous so that it can be solved by adding the usual complementary

SHO solution $x_c = A \cos \omega t + B \sin \omega t$ with $\omega = \sqrt{k/m}$ to the particular solution $x_p = \dot{F}t/k$. Fitting the values of the two integration constants A and B to the initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$, one gets

$$x(t) = \frac{\dot{F}}{k} \left(t - \frac{1}{\omega} \sin \omega t \right) \quad (3)$$

which interestingly also has the property that $a(0) = 0$, a counter-example to the argument one sometimes hears that a ball at the top of its trajectory cannot have zero acceleration “because then it would never start to move again.” Here we have an example of an object that has both zero velocity and acceleration at $t = 0$ and yet begins to move in the next instant because it has nonzero jerk $\dot{a}(0) = \dot{F}/m > 0$.

Alternatively, Eq. (2) can be solved numerically by the Euler-Cromer method. Let F ramp up from 0 to F_{\max} in steps of $\Delta F = F_{\max}/N$ while t increases from 0 to t_{\max} in steps of $\Delta t = t_{\max}/N$, where say $N = 100$ is the number of iterative steps. Suppose $m = 10$ kg, $g = 9.8$ m/s², $k = 50$ kN/m (meaning the upper string stretches about 2 mm when the weight is hung from it), and that the first string to attain a tension of 200 N will break. One can put $F_{\max} = 200$ N, provided one keeps in mind that F will not reach this value if the upper string breaks. The iteration sequence is now as follows and easily done in Excel:

1. Initialize $F_0 = 0$, $x_0 = 0$, $a_0 = 0$, $v_0 = 0$, and $T_0 = mg$. Choose some value for t_{\max} .
2. Update the applied force to $F_n = n\Delta F$ at step $n = 1, 2, 3, \dots$
3. Update the velocity of the ball to $v_n = v_{n-1} + a_{n-1}\Delta t$ from basic kinematics.
4. Update the spring stretch to $x_n = x_{n-1} + v_n\Delta t$ which is Cromer’s improvement to Euler.
5. Update the acceleration of the ball to $a_n = (F_n - kx_n)/m$ according to Eq. (2).
6. Update the tension in the upper string to $T_n = mg + kx_n$ from Eq. (1).

The running time is not directly needed, but can be calculated for plotting purposes as $n\Delta t$. We now loop through steps 2–6 until either the tension in the upper string, T_n , or that in the lower string, F_n , exceeds 200 N. The results agree by eye exactly with the analytic solution by substituting Eq. (3) into (1) to get the tension in the upper string, and computing $\dot{F}t$ to get the tension in the lower string.

Some results are plotted on the next page. Each graph is labeled with the rate of application of the force, $\dot{F} = F_{\max}/t_{\max}$. The red curve that starts at the origin denotes the tension in the lower string and the blue curve beginning at $mg = 98$ N refers to the upper string. We see that if we apply the force slowly, the upper string breaks first, after some interesting driven oscillations, since its tension is always on the order of mg larger than that in the lower string. On the other hand, as the force becomes more impulsive, the lower curve crosses the upper one, so that the lower string does indeed break first; one sees that the tension in the upper string is fairly flat initially in that case, reflecting the fact that the ball does not have sufficient time to move appreciably. For the parameters listed above, the crossover from one regime to the other (when both strings break simultaneously) occurs when the force is applied in $t_{\max} = 27$ ms, corresponding to $\dot{F} = 7.4$ kN/s.

