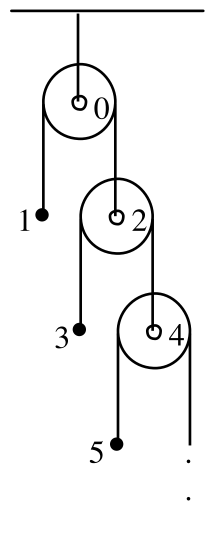
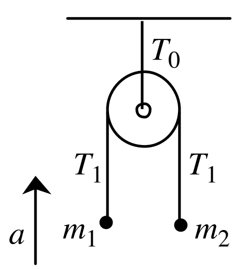
Infinitely-Compound Unit-Mass Atwood Machine—C.E. Mungan, Fall 2019

*reference: AJP **55**, 514 (1987)*

Consider the chain of Atwood machines sketched below, where the point masses hanging from the left side of each ideal pulley have unit mass. (An ideal pulley is massless and has a frictionless axle. In addition, the strings are massless, cannot stretch, and do not slip on the perimeter of the pulley.) The chain continues to infinity in a uniform gravitational field *g*. The problem is to find the upward acceleration *a* of point mass 1.



Denote the upward tensions on the point masses as T_1, T_3, T_5, and so on. Also label the tension in the string attaching pulley 0 to the ceiling as T_0. Since each pulley is ideal, the tension is the same in the string on both of its sides. Thus T_1 is the upward tension on pulley 2. The downward tensions on pulley 2 are T_3 from the left segment and T_3 from the right segment of the string wrapped around its top half circumference. However, since pulley 2 is massless, the net force on it must be zero, so that $T_3 = \frac{1}{2}T_1$. Likewise $T_5 = \frac{1}{2}T_3 = \frac{1}{4}T_1$ and so on. Thus, starting from a significant tension in the upper strings, the tension on any pulley far down the chain must approach zero. How can that be true given that there remains an infinite sequence of pulleys yet lower down that it is attached to? Simple: the pulleys and masses far along the sequence must be approaching freefall, so that they become weightless. That means there is only a finite *effective* mass *m* that pulley 0 needs to support on its right-hand side. By computing *m* we can find *a*. For this purpose, recall the standard analysis of the simple Atwood machine shown next.



The net clockwise force rotating the system of two hanging masses is $(m_2 - m_1)g$ and it accelerates a total mass of $m_1 + m_2$ so that

$$a = \frac{m_2 - m_1}{m_1 + m_2} g . \tag{1}$$

On the other hand, Newton's second law applied to mass 1 alone is

$$T_1 - m_1 g = m_1 a \tag{2}$$

so that

$$T_0 = 2T_1 = 2m_1(g + a) = 2m_1 g \frac{2m_2}{m_1 + m_2} = \frac{4m_1 m_2}{m_1 + m_2} g . \tag{3}$$

Thus the upper string appears to be supporting an effective weight of *mg* where

$$m = \frac{4m_1 m_2}{m_1 + m_2} . \tag{4}$$

If $m_1 = 1$, this formula becomes

$$m = \frac{4m_2}{1 + m_2} . \tag{5}$$

Now imagine a long chain of pulleys as in the first diagram, except suppose that it terminates after some large number of pulleys. Replace the bottom-most pulley (along with its two hanging masses) by an effective mass *m* with $m_2 = 1$ substituted into Eq. (5) to get $m = 2$. (That result makes sense: the bottom-most pulley is rotationally balanced and so it behaves just like a hanging double-unit mass.) Now rename that *m* as $m_2 = 2$ and apply Eq. (5) to the next higher pulley to obtain $m = 8/3$. Continue this way up the chain to pulley 2. Let the effective mass of it and all lower pulleys be designated as *m*. But for an *infinite* chain of pulleys (with an assumed finite effective mass), adding one more pulley to the chain cannot change the effective mass of the whole set. In that case, Eq. (5) implies

$$m = \frac{4m}{1 + m} \quad \Rightarrow \quad m = 3 . \tag{6}$$

Putting $m_2 = 3$ and $m_1 = 1$ into Eq. (1) gives the final answer of

$$\boxed{a = \tfrac{1}{2} g} . \tag{7}$$

Appendix A

We can use these same ideas to find the upward acceleration a of mass 1 for finitely-compound unit-mass Atwood machines. Here are the first few cases.

1 pulley: Equation (1) implies $a = 0$ because the pulley is rotationally balanced.

2 pulleys: Substitute $m_1 = 1$ and $m_2 = 2$ into Eq. (1) to find $a = \frac{1}{3}g$.

3 pulleys: Substitute $m_1 = 1$ and $m_2 = \frac{8}{3}$ into Eq. (1) to find $a = \frac{5}{11}g$.

4 pulleys: Equation (5) becomes

$$m_2 = \frac{4(8/3)}{1+8/3} = \frac{32}{11} \tag{8}$$

which, when substituted into Eq. (1) along with $m_1 = 1$, leads to $a = \frac{21}{43}g$.

We can see that a for these four cases is progressively approaching $a = \frac{1}{2}g$.

Appendix B

Seth Rittenhouse came up with an elegant method to get Eq. (7) more rapidly. Suppose that the upward acceleration of mass 1 is a in earth's frame of reference (which we will call the unprime frame) resulting from tension T upward on it and the gravitational field g pulling downward on it. Now consider the motion of mass 3 in the frame of reference of pulley 2 (which we will call the primed frame). Suppose it accelerates upward at a' due to tension T' upward and gravitational field g' downward. Since pulley 2 is accelerating downward at a, the equivalence principle implies that $g' = g - a$. But since the chain of pulleys is infinite, it must be the case that the ratio of the upward acceleration of the point mass to the downward pull of gravity must be the same in the two frames of reference,

$$\frac{a}{g} = \frac{a'}{g'} \quad \Rightarrow \quad a' = \frac{g-a}{g} a. \tag{9}$$

But Newton's second law applied to mass 1 implies

$$T - mg = ma \quad \Rightarrow \quad T = m(g+a) \tag{10}$$

as in Eq. (2), whereas applied to mass 2 it becomes

$$T' - mg' = ma'. \tag{11}$$

Substituting $T' = T/2$ (as discussed above), $g' = g - a$, and Eqs. (9) and (10) into (11) results in the quadratic equation

$$2a^2 + ga - g^2 = 0 \tag{12}$$

whose positive root is Eq. (7). Inserting that solution into Eq. (9) implies $a' = g/4$ (so that we see how the relative accelerations cascade down the chain of pulleys), which means mass 3 accelerates downward at $a - a' = g/4$ in earth's frame of reference. We also see that $T = 3mg/2$ so that $T' = 3mg/4$ (which thus approaches zero as we go down the chain, as remarked near the beginning of this document).