**Integer Solutions for a Ratio of Two Integer Polynomials—C.E. Mungan, Spring 2020**

Suppose that $A$, $B$, and $N$ are given integers. Find all integer solutions $(n,m)$ to the equation

$$\frac{n^N - A}{n - B} = m\,. \tag{1}$$

In particular, find the largest and smallest possible values of $n$ that solve it.

By long division, one finds that Eq. (1) is equal to

$$n^{N-1} + Bn^{N-2} + B^2 n^{N-3} + \cdots + B^{N-1} + \frac{B^N - A}{n - B} = m\,. \tag{2}$$

We conclude that $B^N - A$ must be divisible by $n - B$. Write out all the integer factors of $B^N - A$, including itself and all positive and negative values. For example, the twelve integer factors of 12 are $\pm\{1,2,3,4,6,12\}$. We find the solutions by requiring any of these factors to be equal to $n - B$. Thus the solutions for $n$ are equal to any of these factors plus $B$. The largest and smallest solutions are therefore given by

$$n = B \pm (B^N - A)\,. \tag{3}$$

The only limitation to this general solution occurs if $B^N = A$. In that case, the polynomials are divisible for <u>any</u> $n$ and there are no largest and smallest possible values of $n$.

As an example, suppose Eq. (1) is

$$\frac{n^3 + 2}{n + 4} = m\,. \tag{4}$$

Then we have $A = -2$, $B = -4$, and $N = 3$. The factors of $B^N - A = -62$ are $\pm\{1,2,31,62\}$ for a total of 8 solutions given by

| factor | $n$ | $m$ |
|---|---|---|
| 62 | 58 | 3147 |
| 31 | 27 | 635 |
| 2 | –2 | –3 |
| 1 | –3 | –25 |
| –1 | –5 | 123 |
| –2 | –6 | 107 |
| –31 | –35 | 1383 |
| –62 | –66 | 4637 |

and we see that the largest and smallest values of $n$ are $-4 \pm 62$. Thanks go to Bob Siddon for the idea behind this method of solution.