Maximum Bob Height of an Interrupted Pendulum—C.E. Mungan, Summer 2006

A simple plane pendulum consists of a point mass m connected to a pivot at the origin (0,0) by an ideal string of length L. Let the x-axis point to the right and the y-axis downward. Suppose that a peg, located directly below the pivot at position $y_\mathrm{p} < L$, interrupts the swing of the pendulum. That is, the bob moves in a circular arc of radius L until the string is vertical. The bob then proceeds around the peg in a smaller circular arc of radius $r = L - y_\mathrm{p}$. We suppose that the string is initially not wrapped around the peg but passes to the left of it, and that the bob is pulled back to the left by an angle θ_i with respect to the $+y$ axis and then released from rest. The problem consists in finding the angle θ_max (to the right of the $+y$ axis) at which the bob attains its maximum height after passing underneath the peg.

The initial coordinates of the bob are $(x_\mathrm{i}, y_\mathrm{i})$ where $x_\mathrm{i} = -L\sin\theta_\mathrm{i} < 0$ and $y_\mathrm{i} = L\cos\theta_\mathrm{i} > 0$ assuming that the initial angle is sensibly restricted to the range $0 < \theta_\mathrm{i} < \pi/2$. The initial height of the bob above the peg is $h_\mathrm{i} = y_\mathrm{p} - y_\mathrm{i}$. If $h_\mathrm{i} \le 0$ then energy conservation implies that the bob rises to the same final height as its initial height, so that

$$\cos\theta_\mathrm{max} = -h_\mathrm{i}/r\,. \tag{1}$$

On the other hand, as noted after Eq. (3) below, if the bob starts out with a height above the peg that is greater than or equal to 1.5r, then the bob will smoothly wind itself up in a circular spiral around the peg, so that $\theta_\mathrm{max} = \pi$. In the remainder of this document, let us consider the intermediate case where $0 < h_\mathrm{i} < 1.5r$. This implies that there is some angle θ between 90 and 180 degrees at which the string first goes slack, because the speed of the ball υ_0 at that point is too small to sustain the centripetal acceleration required for a nonzero string tension. (Define $\theta_0 \equiv \pi - \theta$. Note that θ_0 is the angle between 0 and 90 degrees at which the bob is instantaneously traveling leftward relative to the horizontal.) The radial component of Newton's second law at this point of zero tension is

$$mg\cos\theta_0 = m\frac{\upsilon_0^2}{r} \quad\Rightarrow\quad \frac{1}{2}m\upsilon_0^2 = \frac{1}{2}mgr\cos\theta_0\,. \tag{2}$$

But conservation of mechanical energy between this point and the bob's initial point implies

$$mgh_\mathrm{i} = \frac{1}{2}m\upsilon_0^2 + mgr\cos\theta_0 \quad\Rightarrow\quad \cos\theta_0 = \frac{2h_\mathrm{i}}{3r} \tag{3}$$

after substituting Eq. (2). Note that $\theta_0 \to 0 \Rightarrow \theta \to \pi = \theta_\mathrm{max}$ as $h_\mathrm{i} \to 1.5r$.

Beyond this point, the bob will follow a parabolic trajectory with range R and maximum height H which are found from the usual equations of kinematics. The time to travel in freefall from the string-slackening angle θ to the angle θ_max (at which the maximum height H is attained) is

$$0 = \upsilon_0\sin\theta_0 - gT \quad\Rightarrow\quad T = \frac{\upsilon_0\sin\theta_0}{g} \tag{4}$$

(temporarily redefining upward to be positive). Therefore

$$H = (\upsilon_0 \sin\theta_0)T - \frac{1}{2}gT^2 = \frac{\upsilon_0^2 \sin^2\theta_0}{2g} = \frac{1}{2}r\cos\theta_0 \sin^2\theta_0 \tag{5}$$

and

$$\frac{R}{2} = (\upsilon_0 \cos\theta_0)T = \frac{\upsilon_0^2 \cos\theta_0 \sin\theta_0}{g} = r\cos^2\theta_0 \sin\theta_0 \tag{6}$$

where Eq. (2) was used to eliminate υ_0^2 in the last step of each of these equations. Relative to the peg, we therefore see that the bob will attain its maximum height when its coordinates are

$$x_{\max} = r\sin\theta_0 - \frac{R}{2} = r\sin^3\theta_0 \tag{7}$$

and

$$h_{\max} = r\cos\theta_0 + H = r\cos\theta_0(1 + \tfrac{1}{2}\sin^2\theta_0). \tag{8}$$

The reader may wish to check that $x_{\max}^2 + h_{\max}^2 < r^2$ for all $0 < \theta_0 < \pi/2$, as required for the string to remain slack. We conclude that

$$\theta_{\max} = \frac{\pi}{2} + \tan^{-1}\frac{h_{\max}}{x_{\max}} = \frac{\pi}{2} + \tan^{-1}\left[\cot\theta_0\left(\tfrac{1}{2} + \csc^2\theta_0\right)\right]. \tag{9}$$

Substituting for θ_0 from Eq. (3), I plot $\theta_{\max}$ as a function of h_i / r below using Eq. (9) for positive h_i and Eq. (1) for negative h_i. (Note that $\theta_{\max} = \pi$ for all $1.5 < h_i / r < L/r - 1$.)